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Dynamics of Collectives with Opinionated Agents: The Case of Scrambling Connectivity.

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Abstract— **We investigate the problem of social/opinion networks of autonomous agents, each of which has a personal view that follows an independent dynamic behavior and it communicates with their neighbors under a cooperative algorithm. We apply mathematics from non-negative matrix theory. We investigate agents with identical and non-identical views and we establish sufficient conditions for stability and convergence to a common synchronized opinion.**

I. INTRODUCTION

The dynamics of collectives in social sciences were originally considered to be one of the most general and fundamental problems of sociology. They are nowadays in the center of attention of the applied science and the engineering communities. The development and popularity of web-based social networks, for example, has attracted the attention of scientists and engineers [6]. The need to understand the structure and dynamics on such networks sheds new light on the classic problem of human action and motivation.

On a totally different vein, recent advances in the mathematics of biologically inspired networks of autonomous agents (ants or fishes or even birds) introduce a whole family of rigorous models that generate global self-organizing behavior out of local interactions, [3], [5], [13], [12], [14].

As society is constituted by human individuals which interact with each other in a primarily local manner, any coordination of opinions on a specific matter or aspect of life may occur after, and due to, exchanging information in a cooperative rather than competitive attitude. The formation of social groups and structures in a society may occur from individuals who share similar material, emotional and mental needs and weakly interact with each other to converge to a synchronized state or strongly interact with each to overcome their individualistic views and collectively converge to forming the group. In any case, social sciences treat individuals as autonomous intelligent entities with independently generated and evolving opinions, i.e. opinions the state of which does not change in time exclusively as a result of local interaction but it is also a function of internal dynamics of the individual.

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This crucial distinction between human and animal social groups places the problem of convergence to consensus in a new perspective. Hence, it is of interest to study the long term behavior of opinionated agents, (i.e. agents evolving their opinion in an autonomous way) who seek to promote solidarity with respect to their opinion, within their group, using cooperative algorithms. In other words, we need to understand how the individual (internal) dynamics affect the collective behavior. We will see that in some cases they enhance it, in some cases they avert it and in many cases we observe the emergence of new dynamics.

A. Related Literature & This Work

The study of dynamics of interacting entities is by no means new. It has been the subject of interdisciplinary scientific fields, from Sociology, to Chemistry, Physics and Applied Mathematics [1], [3], [7], [8], [10], [16] with various rigorous frameworks to have contributed to the subject from microscopic sets of differential equations to meso and macroscopic, evolutionary partial differential equations of distributions.

The present work takes a first step towards merging the mathematics of flocking and consensus networks for continuous time systems that were developed in [12], [13], [14] with dynamics on graphs, under the assumption that individuals sustain their own way of thinking. This means that agents evolve their opinions independently in addition to the coupling effect motivated by the consensus averaging. This new feature causes new types of long-term solutions other than common equilibrium.

Although the mathematical models to be developed here are compatible with several distinct interpretations the authors aim to sustain the social interaction point of view. The results are to be interpreted in this manner. This point will be revisited in the last section of this paper.

II. NOTATION & DEFINITIONS

The underlying mathematical notions are standard ordinary differential equation theory and theory of graphs and non-negative matrices [4], [9], [11]. A function a is a member of $C^l(D, R)$ if it is defined in D, takes values in R and has $l \geq 0$ continuous derivatives. The differential operator $\frac{d}{dt}$ will denote the generalized right Dini-derivative. Throughout this paper $t_0 \in \mathbb{R}$ is assumed an arbitrary but fixed initial time.

We consider a population of $N < \infty$ individuals, $[N] = \{1, \ldots, N\}$ each of which possesses an opinion, quantified by a number and generally denoted as $x_i \in \mathbb{R}$, $i \in [N]$. For every agent $i \in [N]$ we define its neighborhood of

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effect $N_i \subset [N]$, i.e. all other agents that affect i. This communication scheme is best represented through a directed graph $G = ([N], E)$ where now $[N]$ is the set of vertices and $E = \{(i,j) : i \neq j \in [N]\}$ is the set of connections between them. In view of the graph theoretic setting $N_i = \{j \in [N] : (i,j) \in E\}$. The connectivity weight between two vertices i and j is denoted by a_{ij}, it is a non-negative number and it signifies the rate with which j affects i. In this work, we will consider non-negative, time-dependent weights. In terms of graph theory, we can define a time-varying directed communication graph $G_t = ([N], E_t)$ so that $(i,j) \in E_t$ if and only if $a_{ij}(t) > 0$. Next, we define the *spread* of the state vector $\mathbf{x} \in \mathbb{R}^N$ as

$$S(\mathbf{x}) = \max_i x_i - \min_i x_i$$

The spread S is a pseudo-metric: It is non-negative and it satisfies the triangular inequality, but $S(\mathbf{x}) = 0$ implies that $\mathbf{x}$ has identical elements. The dynamical behavior of $S(\mathbf{x})$, for $\mathbf{x} = \mathbf{x}(t),\ t \geq t_0$, a solution of the models under study will be of central interest throughout the paper.

Definition 2.1: We say that the population is asymptotically synchronized if $S\big(\mathbf{x}(t)\big) \to 0$ as $t \to \infty$.

III. PRELIMINARIES

Each agent $i \in [N]$ is characterized by its state x_i and the dynamics this state obeys. On the internal dynamics part each individual evolves its state according to the initial value problem

$$\dot{z} = f(t, z),\ t \geq t_0 \tag{1}$$

with solution $z = z(t, t_0, \xi),\ t \geq t_0$ and $z(t_0) = \xi$.

Assumption 3.1: There exist $\overline{C} > 0$ such that $f(t,x)$ is defined in $[t_0, \infty) \times (-\overline{C}, \overline{C})$, it takes value in a uniformly bounded subset of $\mathbb{R}$, it is continuous in t and it has continuous first derivative in x. Also there exists a positive $C < \overline{C}$ and an open subset of $(-C, C)$, W, so that $\xi \in W$ implies

$$|z(t, t_0, \xi)| \leq C$$

for $t \geq t_0$.

The theory of differential equations assures that the forward limit set $\omega(\xi)$ is non-empty, compact, connected subset of $\mathbb{R}$ and that for every $\phi^0 \in \omega(\xi)$ there exists a solution $\phi(t, t_0, \phi_0)$ of the above initial value problem. This is the minimum condition we demand for the internal behavior. We classify two types of populations:

1) Homogeneous Dynamics: Agents with identical f. In this case our objective will be to derive sufficient conditions for asymptotic convergence to a common, synchronized solution. This is feasible because of the identical dynamics. It is noted that although f is the same for every agent, they can still develop different opinions as f can attain multiple equilibrium states.
2) Heterogeneous Dynamics: Agents with non-identical f. Here the objective will be to estimate the distance of opinions based on the strength of the coupling interactions as well as the difference of $f^{(i)}$. We will assume that the functions $f^{(i)}$ can be parametrized by a variable ϵ_i so that $f^{(i)}(t, x_i) = f(\epsilon_i, t, x_i)$ and that $f^{(i)}$ depends smoothly on ϵ_i.

The second part of the overall dynamics concerns the coupling function between the agents' opinions. This is assumed to be linear and to involve the difference between the interacting agents states. The consideration is motivated by the literature in consensus dynamics so that the tools from non-negative matrix theory can be applied for the rigorous analysis. The interactions between the i and j are modeled through non-negative functions $a_{ij}(t)$, under the following condition:

Assumption 3.2: $\forall\ i \neq j \in [N]$, $a_{ij} \in C^0([t_0, \infty), \mathbb{R}_+)$.

This condition expresses the cooperative mode of the interaction dynamics and it will be assumed throughout the paper.

IV. THE MODELS

A finite population of opinionated agents gets networked through a communication graph G_t and updates their opinion as a combination of the internal dynamics and the local interactions. We will distinguish between two cases as noted

A. Homogeneous Dynamics

In the first case, the internal dynamics evolve under the same rule for all agents. This is modeled through a common function f and the model in question is:

$$\dot{x}_i = f(t, x_i) + \sum_{j \in N_i} a_{ij}(t)(x_j - x_i),\ t \geq t_0 \tag{2}$$

where $i \in [N]$ and $x_i(t_0) = \xi_i^0$ are the index of the agent and the initial condition, respectively. The solution of the overall system is denoted by $\mathbf{x}(t, t_0, \boldsymbol{\xi}),\ t \geq t_0$. The system (2) can be used to model populations where individuals can have a different personalized view on a subject but for example very similar (here identical) way of thinking. Distinct premises or experiences towards the subject in question, more likely than not, lead to distinct attitudes and in the long run shape a different opinion. However the logic, i.e. the reasoning and the methods used to derive this conclusion may, however, be very similar or even identical. Such a scenario justifies models like (2).

B. Heterogeneous Dynamics

The case of non-identical reasoning or different logic is modeled through the system

$$\dot{x}_i = f(\epsilon_i, t, x_i) + \sum_{j \in N_i} a_{ij}(t)(x_j - x_i),\ t \geq t_0 \tag{3}$$

where $i \in [N]$ and $x_i(t_0) = \xi_i^0$ are the agent and the initial condition, respectively. It will be assumed that for every admissible ϵ, $f(\epsilon, \cdot, \cdot)$ satisfies Assumption 3.1 and that $f(\cdot, t, x) \in C^1$.

The regime under which both (2) and (3) evolve can be characterized by a time-dependent directed communication graph G_t as it was discussed in §II. We are interested in connectivity regimes that sustain the scrambling property, even in some averaging sense: What we essentially ask is that for every two vertices i and i' there exists at least a third one j such that $a_{ij}(t) > 0$ and $a_{i'j}(t) > 0$. Such non-negative matrices are known in the literature as scrambling matrices [4].

C. Preliminaries

We begin the discussion with a couple of crucial observations. Since (2) is only a special case of (3) the observations are based only on the latter system.

Proposition 4.1: Consider (3) under Assumption 3.1 to hold for all IVPs $\dot{z} = f^{(i)}(t,z)$ with $z(t,t_0,\xi_i)$, as well as under Assumption 3.2. Let $\xi_i \in W$, then the solution $\mathbf{x}(t,t_0,\boldsymbol{\xi})$ of (3) satisfies

$$|x_i(t,t_0,\xi_i)| \le C, \qquad t \ge t_0$$

Proof: Assume that the first time, over all agents, agent i hits C from the left to the right is t'. It then must hold that $x_i(t') = C$ and $\dot{x}_i(t') > 0$. This is a contradiction in view of the Assumptions taken, the fact that $x_j(t') \le x_i(t')$ and the form of (3). ■

The main remark from the above result is that so long as the couplings a_{ij} satisfy Assumption 3.2, the stability of the overall network is not critically affected.

D. Analysis

The aim of this subsection is the rigorous investigation of the asymptotic dynamics of the initial value problems (2) and (3). We will establish sufficient conditions for convergence to synchronization for the former network and generalized boundedness estimates for the latter one.

1) The Homogeneous Network: Although we have identical internal dynamics, different initial values can yield to different trapping regions and limit sets. The objective is to derive a simple sufficient condition that characterizes the strength of the interaction couplings and the internal dynamics so that the states converge to a synchronized behavior. For this we will need to define the following quantity:

$$\tau(t) = \min_{i,i'} \left\{ \sum_{k \neq i,i'} \min\left\{a_{ik}(t), a_{i'k}(t)\right\} + a_{i'i}(t) + a_{ii'}(t) \right\}. \tag{4}$$

Note that τ depends solely on the communication network between the agents and not on the internal dynamics.

Theorem 4.2: Consider the solution $\mathbf{x}(t,t_0,\boldsymbol{\xi})$, $t \ge t_0$ of the IVP (2) such that $\xi_i \in W$, $\forall i \in [N]$, and let Assumptions 3.1 and 3.2 hold. If

$$\int_{t_0}^{\infty} \tau(s) - \left[\max_{y_1,y_2 \in C} \int_0^1 f'\big(s, wy_1 + (1-w)y_2\big)\,dw\right] ds = \infty,$$

then solutions converge to a synchronized state according to Definition 2.1.

Proof: Consider the solution $\mathbf{x}$ and pick $m > -\sup_t \max_i \sum_j a_{ij}(t) + \sup_t \max_{x\in[-C,C]} f(t,x) + \sup_t \max_{i,j} a_{ij}(t)$. By assumption, there exists such a finite m. Then we can write

$$\dot{x}_i = -mx_i + (m-d_i)x_i + \sum_j a_{ij}(t)x_j + f(t,x_i) \;\Leftrightarrow$$
$$e^{-mt}\frac{d}{dt}\big(e^{mt}x_i\big) = (m-d_i)x_i + \sum_j a_{ij}(t)x_j + f(t,x_i)$$

Pick $i, i' \in [N]$. Then from the above equation we have:

$$e^{-mt}\frac{d}{dt}\big(e^{mt}(x_i - x_{i'})\big) = \sum_j \tilde{a}_{ij}(t)x_j - \sum_j \tilde{a}_{i'j}(t)x_j$$

where

$$\tilde{a}_{ij}(t) := \begin{cases} m - d_i(t) + \int_0^1 \frac{\partial}{\partial x} f(t, sx_i + (1-s)x_{i'})\,ds, & j = i \\ a_{ij}(t), & j \neq i. \end{cases}$$

Next

$$e^{-mt}\frac{d}{dt}\big(e^{mt}(x_i - x_{i'})\big) = \sum_j w_j(t)x_j$$

for $w_j(t) = \tilde{a}_{ij}(t) - \tilde{a}_{i'j}(t)$. The index for which $w_j(t) > 0$ is denoted by j^+ and the index for which $w_j(t) \le 0$ is denoted by j^-. We remark that $\sum_j w_j(t) \equiv 0$. Set

$$\theta = \sum_{j^+} w_{j^+}(t) = \sum_{j^+} |w_{j^+}(t)| = -\sum_{j^-} w_{j^-}(t)$$
$$= \sum_{j^-} |w_{j^-}(t)| = \frac{1}{2}\sum_j |w_j(t)| = \frac{1}{2}\sum_j |\tilde{a}_{ij} - \tilde{a}_{i'j}|$$

Then

$$e^{-mt}\frac{d}{dt}\big(e^{mt}(x_i - x_{i'})\big) =$$
$$= \theta\left(\frac{\sum_{j^+} |w_{j^+}(t)|x_{j^+}}{\theta} - \frac{\sum_{j^-} |w_{j^-}(t)|x_{j^-}}{\theta}\right) \le \theta\, S(\mathbf{x}).$$

Finally from the identity $|x - y| = x + y - 2\min\{x,y\}$ we deduce that

$$\theta = m + \int_0^1 f'(t, sx_i + (1-s)x_{i'})\,ds - \sum_k \min\{\tilde{a}_{ik}(t), \tilde{a}_{i'k}(t)\}$$

and by the choice of m (large enough) the summation of the minima over k cannot include $a_{ii}(t)$ or $a_{i'i'}(t)$, so we are left with the off-diagonal terms and

$$\theta(t) \le m - \tau(t) + \max_{x_1,x_2 \in C} \int_0^1 f'\big(t, wx_1 + (1-w)x_2\big)\,dw.$$

Finally, for i, i' that maximize $x_i - x_{i'}$ we have

$$\frac{d}{dt}S\big(\mathbf{x}(t)\big) = \frac{d}{dt}\left[e^{-mt}S\big(e^{mt}\mathbf{x}(t)\big)\right] =$$
$$= -mS\big(\mathbf{x}(t)\big) + e^{-mt}\frac{d}{dt}S\big(e^{mt}\mathbf{x}(t)\big)$$
$$\le \left(-\tau(t) + \max_{x_1,x_2\in C}\int_0^1 f'\big(t, wx_1 + (1-w)x_2\big)\,dw\right) S\big(\mathbf{x}(t)\big)$$

and the proof is concluded in view of Gronwall's Lemma. ■

Theorem 4.2 simply relates the coupling effect between the agents with the internal dynamics of the agents opinion. Indeed the sign of $\max_{y_1,y_2\in C}\int_0^1 f'\big(s, wy_1 + (1-w)y_2\big)\,dw$

is a characterization of the stability properties of the region C. For example, if the dynamics evolve in the vicinity of a fixed point $x = p$ attractor then a sufficient asymptotic stability would be $f'(p) < 0$ and any coupling effects are obsolete. On the other hand if C includes two or more attracting sets then coupling is necessary for synchronization. The stated condition in the Theorem is then sufficient for synchronization.

2) The Heterogeneous Network: A crucial assumption in the preceding models is that, agents enjoy identical internal dynamic evolution. This although realistic in societies with like-minded (or programmed) entities it may be a strict simplification in other real-world phenomena. For humans no matter how rational they are assumed to be, it is highly unlikely that their views on a subject could be identical. Different mentalities, regardless the coupling strength, is not possible to lead to identical synchronization. The model we consider in this section assumes different functions for the internal dynamic behavior. In particular, we will assume that f can be parametrized by a parameter ϵ, and that f depends smoothly on this parameter. Our model now is the initial value problem (3). Define for the sake of brevity

$$\kappa(t) = \tau(t) - \max_{y_1,y_2\in C} \int_0^1 f'\big(t, qy_1 + (1-q)y_2\big)\, dq$$

$$\lambda(t) = \max_{i,i'\in[N]} \max_{y\in C} \int_0^1 \frac{\partial}{\partial\epsilon} f\big(q\epsilon_i + (1-q)\epsilon_{i'}, t, y\big)\, dq(\epsilon_i - \epsilon_{i'})$$

where $\tau(t)$ is as (4).

Theorem 4.3: Consider the solution $\mathbf{x}(t, t_0, \boldsymbol{\xi}),\ t \geq t_0$ of the IVP (3) such that $\xi_i^0 \in W, \forall\, i \in [N]$, and let Assumption 3.1 and 3.2 hold. Then

$$\sup_{t\geq t_0} S\big(\mathbf{x}(t)\big) \leq \sup_{t\geq t_0} \left\{ e^{-\int_{t_0}^t \kappa(s)\, ds} S(\boldsymbol{\xi}) + \int_{t_0}^t e^{-\int_s^t \kappa(w)\, dw} \lambda(s)\, ds \right\}.$$

In addition, if there exist constants $\underline{\kappa}$ and $\overline{\lambda}$ such that $\kappa(t) \geq \underline{\kappa} > 0$ and $\lambda(t) < \overline{\lambda}$ then $\lim_{t\to\infty} S\big(\mathbf{x}(t)\big) \leq \frac{\overline{\lambda}}{\underline{\kappa}}$.

Proof: Following Theorem (4.2), we have

$$\dot{x}_i = -mx_i + (m - d_i)x_i + \sum_j a_{ij}(t)x_j + f(\epsilon_i, t, x_i) \ \Leftrightarrow$$

$$e^{-mt}\frac{d}{dt}\big(e^{mt}x_i\big) = (m - d_i)x_i + \sum_j a_{ij}(t)x_j + f(\epsilon_i, t, x_i)$$

Pick $i, i' \in [N]$. Then

$$e^{-mt}\frac{d}{dt}\big(e^{mt}(x_i - x_{i'})\big) = \sum_j \big(\tilde{a}_{ij}(t) - \tilde{a}_{i'j}(t)\big)x_j +$$

$$+ \int_0^1 \frac{\partial}{\partial\epsilon} f(s\epsilon_i + (1-s)\epsilon_{i'}, t, x_i)\, ds(\epsilon_i - \epsilon_{i'})$$

where

$$\tilde{a}_{ij}(t) := \begin{cases} m - d_i(t) + \int_0^1 \frac{\partial}{\partial x} f(t, sx_i + (1-s)x_{i'})\, ds, & j = i \\ a_{ij}(t), & j \neq i. \end{cases}$$

as before. Then

$$S\big(\mathbf{x}(t)\big) \leq e^{-\int_{t_0}^t \kappa(s)\, ds} S(\boldsymbol{\xi}) + \int_{t_0}^t e^{-\int_s^t \kappa(w)\, dw} \lambda(s)\, ds$$

for κ and λ defined above. The final conclusion is reached with an elementary calculation. ■

V. Simulation Examples

Let us consider the network of $N = 6$ coupled agents with the connectivity scheme that is depicted in Fig. 1. We will examine three scenarios. In the first scenario, we have identical autonomous opinions with different initial positions and uniform coupling weights. In the second scenario the population has homogeneous dynamics (common f) which will be time-dependent. Again strong enough coupling will lead to synchronized behavior. In the third, the scenario of heterogeneous dynamics is considered and the result of Theorem 4.3 is investigated. In all three scenarios $t_0 = 0$ and f are appropriately chosen so that Assumption 1 is satisfied for appropriate $\underline{C}$, $\overline{C}$ and initial values in W.

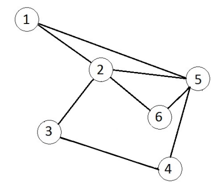

Fig. 1. The simulation example communication network. Every edge signifies the existence of directed coupling between the corresponding vertices. It can be easily verified that the connectivity corresponds to a scrambling nonnegative matrix.

A. Autonomous homogeneous. Constant couplings.

Here $f(t, x) = \sin(x)$ and $a_{ij} \neq 0$ implies $a_{ij} \equiv a$. Figure 2 illustrates the simulations run with the `ode45` routine. The initial values are $\boldsymbol{\xi} := [-8, 2, -1, 0.5, 9.2, -3]^T$. According to Assumption 3.1 the uncoupled solutions are uniformly boundend and so is $\frac{\partial}{\partial x} f(x)$. So all assumptions are satisfied and Theorem 4.2 holds.

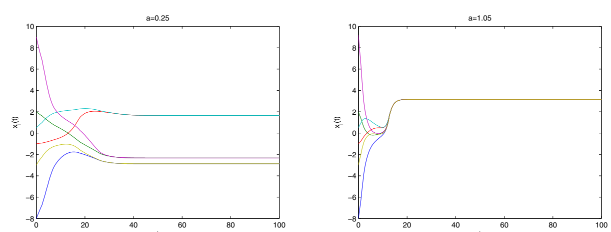

Fig. 2. Simulation of the first scenario. For weak coupling and initial data that converge to different equilibria, there is not convergence to a common value. Here $a = 0.25$ does not satisfy the condition of Theorem 4.2. Stronger coupling $a = 1.05$ does satisfy the condition of Theorem 4.2 and it leads to consensus.

B. Non-autonomous homogeneous. Time-Dependent couplings.

Here $f(t, x) = 20\sin(10t)\sin(0.9x)$ The adjacency matrix $A = [a_{ij}(t)]$ where for all $j \in N_i$, $a_{ij} = a(1.05 + \sin(t))b_{ij}$, $b_{ij} \in (1, 2)$ chosen randomly and $a > 0$. In this case (4) yields the estimate $\tau(t) > a(1.05 + \sin(t))$. For a small enough there is not synchronization according to simulation results presented in Fig 3. The initial data here is chosen again appropriately so that the uncoupled network to exhibit bounded solutions and Assumption 3.1 to hold.

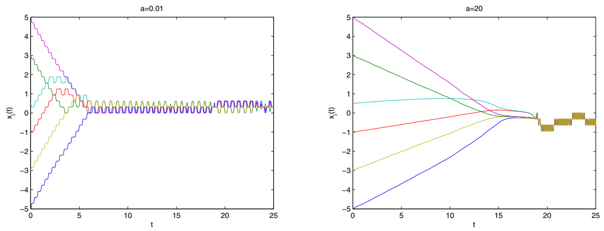

Fig. 3. Simulation of the second scenario. For weak coupling and initial data that converge to different equilibria, there is not convergence to a common value. The situation changes for strong coupling. Due to the non-autonomous nature of the dynamics, the agents converge to a non-constant synchronized solution.

C. Heterogeneous. Time-Dependent couplings.

In the event of non-identical opinionated agents we cannot have complete synchronization in the sense of Definition 2.1. We will test the estimate on the spread following Theorem 4.3 under the assumption that $f(\epsilon_i, t, x) = 10\sin(10t)\sin(\epsilon_i x)$ and $\epsilon_1 = 0.5, \epsilon_2 = 0.1, \epsilon_3 = 0.6, \epsilon_4 = 0.3, \epsilon_5 = 0.2, \epsilon_6 = 0.4$. The couplings are as in the second scenario and numerical calculation yields $\overline{\lambda} \leq 5$. The estimate of Theorem 4.3 yields

$$\lim_t S\big(\mathbf{x}(t)\big) \leq 34,$$

a very conservative estimate in comparison to the simulation result presented in Figure 4.

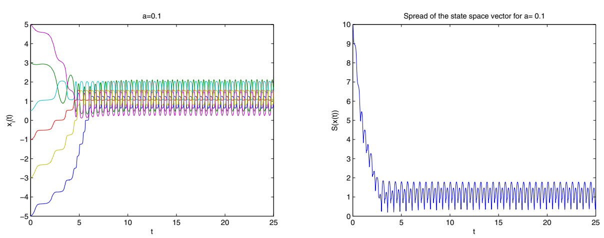

Fig. 4. Simulation of the third scenario. There is no chance for complete synchronization. The stronger the coupling is, the smaller the $S(\mathbf{x}(t))$ becomes, as $t \to \infty$. Here $S(x(t)) < 2$ as $t >> t_0$.

VI. Discussion

A. From scrambling to simply connected networks

The use of non-negative matrix theory methods yields elegant results on the asymptotic network dynamics relating the internal behavior and the coupling forces. A deeper inspection, however, will reveal that the network is asked to be unnecessarily connected for simple convergence to synchronization. Indeed τ is non-zero at instant t if for every two agents there is a third one that affects both of them. A well-known result in the control community, that comes out of the theory of non-negative matrices, is that simply connected networks (where $f \equiv 0$) can converge to a common equilibrium under mild connectivity conditions that characterize the network in a uniform recurrent manner. This argument was exploited in [5] for proving convergence for the Viscek flocking model [15]. In [13], [14] the authors revisited the problem under such mild conditions with the use of the present framework and they managed to establish new results for nonlinear second order (flocking) networks of Cucker-Smale type [3].

The presence of internal dynamics makes the synchronization problem challenging because it essentially destroys the nice monotonicity condition that consensus systems enjoy i.e. the spread of the states is a non-increasing function of time. We wish to keep the discussion within the means of scrambling connectivity and the elegant results it provides. Thus we will provide only a heuristic presentation of how the analysis should proceed under the present context. The main characteristic of the communication network (i.e. the weights a_{ij}) is that there exists $B > 0$ such that the union $\bigcup_t^{t+B} G_t$ corresponds to a simply connected graph. In this case we can write the differential equation of (2) in vector form as follows:

$$\dot{\mathbf{x}} = -mI\mathbf{x} + (mI - D(t))\mathbf{x} + A(t)\mathbf{x} + \mathbf{f}(t, \mathbf{x}) \;\Leftrightarrow$$

$$\mathbf{x}(t) = \int_{t-B}^{t} \Big[P(t,s)\mathbf{x}(s) + e^{-m(t-s)}\mathbf{f}\big(s, \mathbf{x}(s)\big)\Big]\, ds$$

where $[A(t)]_{ij} = a_{ij}(t)$ also known as the adjacency matrix, $D(t) = \text{Diag}[\sum_j a_{ij}(t)]$ is the valency matrix, and

$$[P(t,s)]_{ij} = \begin{cases} e^{-m(t-s)}\big(\delta_{s-(t-B)} + (m - d_i(s))\big), & i = j \\ e^{-m(t-s)} a_{ij}(s), & i \neq j \end{cases}$$

and $\mathbf{f}(t, \mathbf{x}) = \big(f(t, x_1), \ldots, f(t, x_N)\big)^T$. Now we can repeat the argument over the next B time interval as follows

$$\begin{aligned} \mathbf{x}(t) = &\int_{t-B}^{t} P(t,s) \int_{s-B}^{s} P(s,w)\mathbf{x}(w)\, dw + \\ &+ \int_{t-B}^{t} P(t,s) \int_{s-B}^{s} e^{-m(s-w)}\mathbf{f}\big(w, \mathbf{x}(w)\big)\, dw + \\ &+ \int_{t-B}^{t} e^{-m(t-s)}\mathbf{f}\big(s, \mathbf{x}(s)\big)\, ds \end{aligned}$$

It can be easily shown that $\int_{t-B}^{t} P(t,s)\, ds$ is a stochastic matrix and so is $P^{(2)}(t) := \int_{t-B}^{t}\int_{s-B}^{s} P(t,s)P(s,w)\, dwds$ [13]. In fact $P^{(\sigma)}(t)$ is scrambling if σ is chosen large enough and the contraction (averaging) feature emerges. A change of the order of integration and similar steps in the proof of Theorem 4.2 can yield $S(\mathbf{x}(t))$ to satisfy

$$S(\mathbf{x}(t)) \leq \int_{t-\sigma B}^{t} k(t,s) S(\mathbf{x}(s))\, ds,$$

for any appropriate quantity $k(t,s)$ that depends on $\frac{\partial}{\partial x} f$, m and a_{ij}. It can be shown that for large a_{ij}, m can be chosen large enough as well so that $\sup_t \int_{t-\sigma B}^{t} |k(t,s)|\, ds < 1$ and asymptotic stability can be deduced via a standard fixed point theory argument [2].

B. Nonlinear Couplings

The theory is flexible enough to incorporate nonlinear coupling functions. Minor modifications of the present framework can adequately handle networks of type

$$\dot{x}_i = f(t, x_i) + \sum_j g_{ij}(t, x_j) - \sum_j g_{ij}(t, x_i)$$

or of the type

$$\dot{x}_i = f(t, x_i) + \sum_j g_{ij}(t, x_j - x_i).$$

The former category results in networks with positive monotonic functions $g_{ij}(t, \cdot)$ and the second category asks for g_{ij} to preserve the passivity property (see [8], [13] and references therein).

C. Other Extensions - Future Work

The aim of this subsection is to group a number of extensions according to the authors' views cannot be handled within the context of the present theory, yet they are worth mentioning. The study of models of the type (2) or (3) within the means provided by branches of dynamical system theory can shed light upon other features of these systems [17].

1) Connection with the theory of dynamical systems. Lyapunov Exponents & Bifurcations: As the contraction coefficient is a lower bound of the second eigenvalue of linear time invariant consensus networks, to the best of our knowledge the relation between the stability invariant measures for general linear and nonlinear systems, such as the Lyapunov exponents and the contraction coefficient, is still unexplored. In this work a first step towards this direction is taken. On the other hand it would be very interesting to study the coupling strength and the asymptotic behavior in terms of bifurcation theory. What is the bifurcation values that achieve synchronized solutions? What is the interdependence between f, a_{ij} and ξ_i that define bifurcation curves?

2) Synchronization of oscillators and new consensus schemes: The form of (2) and (3) bear great resemblance with clusters of coupled oscillators. Indeed, in the scalar case presented here, even with the very conservative stability conditions imposed here, the dynamics are still much clearer than the multi-dimensional case. The theoretical framework is not directly applicable to the case that the network consists of agents with multi-dimensional internal dynamics, i.e. $x \in \mathbb{R}^l$. Such networks were introduced and studied by the seminal work of Pecora et al. in [10] for the case of chaotic oscillators. There is however a very interesting turn of events if one attempts to naively apply the present context. For example when $l = 2$ and $N = 2$ the resulting systems in vector form are of the following type

$$\dot{x} = -L_{11}x - L_{12}y$$
$$\dot{y} = -L_{21}y - L_{22}x$$

where L_{ij} are 2×2 matrices such that $L_{ij}\mathbf{1} \equiv 0$ (known as graph Laplacians), which illustrate the interdependence between the x and y components of the network at the two different levels $l = 1$ and $l = 2$. These types of networks are completely new and their dynamics are largely unknown even in the elementary time invariant case.

VII. Conclusion

In this paper we extended previous work on agreement networks under the condition that the agents of the network are enhanced with personal independent dynamical behavior. We studied homogeneous and heterogeneous versions of networks and established conditions on the relation between the internal dynamic rules and the communication coupling so that the solutions of the network become asymptotically synchronized for the case of identical internal dynamics or bounded in the case of non-identical internal dynamics. The overall point of view regarding the interpretation of the models was motivated by social networks and dynamics of gossip algorithms.

Regarding the advantages of our approach we remark that it can handle general linear as well as nonlinear dynamics and communication couplings just as easily. Under the assumption of increased connectivity, the method yields elegant and illustrative convergence conditions that can be easily verified. Regarding the disadvantages of our method we point out that the estimates can be very conservative, a point made clear in the Examples section. This is a well-known weakness of the mathematics of non-negative matrices that has been pointed out in the past [13], [14].

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