CONSTRUCTING OPTIMAL WAVELET BASIS FOR IMAGE COMPRESSION

Yan Zhuang [1] *John S. Baras* [2]

[1] Iterated Systems, 3525 Piedmont Road, Seven Piedmont Center, Atlanta, GA 30305. Email: yzhuang@iterated.com.
[2] Institute for Systems Research, University of Maryland, College Park, MD 20742. Email: baras@isr.umd.edu.

ABSTRACT

We study the problem of choosing an image based optimal wavelet basis with compact support for image data compression and provide a general algorithm for computing the optimal wavelet basis. We parameterize the mother wavelet and scaling function of wavelet systems through a set of real coefficients of the relevant quadrature mirror filter (QMF) banks. We further introduce decomposition entropy as an information measure to describe the distance between a given digital image and its projection into the subspace spanned by the wavelet basis. The optimal basis for the given image is obtained through minimizing this information measure. The resulting subspace is used for image analysis and synthesis. Experiments show improved compression ratios due to the application of the optimal wavelet basis and demonstrate the potential applications of our methodology in image compression. This method is also useful for constructing efficient wavelet based image coding systems.

1. INTRODUCTION

The last few years have witnessed extensive research interest and activities in wavelet theory and its applications in signal processing, image processing and many other fields [1, 2]. The most attractive features of wavelet theory are the multiresolution property and time and frequency localization ability. There are many applications of these properties in the fields of signal processing, speech processing and especially in image processing [3, 4, 5, 6].

It is well known that a wavelet system is usually determined by one mother wavelet function whose dilations and shifts span the signal space. Unlike *sin* and *cos* functions, individual wavelet functions are quite localized in frequency and time and they are not unique. Obviously, different wavelets shall yield different wavelet bases. An appropriate selection of the wavelet for signal representation can result in maximal benefits of this new technique. For example, compact wavelets are suitable for approximating discontinuous functions such as images while smooth wavelets are appropriate for solving integral functions to achieve high numerical accuracy. It is reasonable to think that if a wavelet contains enough information about an image to be represented, the wavelet system is going to be simplified in terms of the levels of required resolution. We are interested in finding an image based wavelet basis and applying the resulting wavelet system to improve image compression ratios.

The key to choosing an image based optimal wavelet basis lies in the appropriate parameterization and adequate performance measure in image compression processes. A method was proposed for choosing a wavelet for signal representation based on minimizing an upper bound of the L^2 norm of error [7, 8] in approximating the signal up to a desired scale. Coifman *et al.* derived an entropy based algorithm for selecting the best basis from a library of wavelet packets [9]. We also proposed an information measure based approach for constructing an optimal discrete wavelet basis with compact support in our earlier work on adaptive wavelet neural networks [10] and wavelet basis selection [11]. We shall illustrate the application of our methodology to image compression.

This paper is intended to demonstrate that choosing an image based optimal or suboptimal wavelet basis can improve compression ratios of images rather than to design a complete coding system. In the rest of the paper, we first provide the definition of optimal wavelet basis for a given digital image and parameterize the basis through the corresponding quadrature mirror filter (QMF) banks. We then introduce an algorithm for constructing an optimal wavelet basis. Next, we compare the effects of different mother wavelets on image representation and provide numerical results. Finally, we summarize our conclusions.

2. OPTIMAL WAVELET BASIS

We first introduce a distance measure for optimization purpose. Inspired by the work in [9], we define an additive information measure of entropy type and the optimal basis as the following. We use $\Psi(t)$ to denote the wavelet basis spanned by dilating and shifting mother wavelet denoted by $\psi(t)$.

Definition 2..1 *A non negative map $\mathcal{M}$ from a sequence $\{f_i\}$ to R is called an additive information measure if $\mathcal{M}(0) = 0$ and $\mathcal{M}(\sum_i f_i) = \sum_i \mathcal{M}(f_i)$.*

Definition 2..2 *Let $x \in R^N$ be a fixed vector containing digital image data and $\mathcal{B}$ denote the collection of all orthonormal bases of dimension N, a basis $B \in \mathcal{B}$ is said to be optimal if $\mathcal{M}(Bx)$ is minimal for all bases in $\mathcal{B}$ with respect to the vector x.*

The wavelet system is parameterized through using QMF banks. From the multiresolution property of wavelets due

0-7803-3192-3/96 $5.00©1996 IEEE

to Mallat [12], the scaling function $\phi(t)$ and mother wavelet $\psi(t)$ are expressed as [2]

$$\phi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \phi(2t - k) \quad (1)$$

and

$$\psi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} d_k \phi(2t - k) \quad (2)$$

where the coefficient $\{c_k\}$ and $\{d_k\}$ determine a low pass filter $h_0(k) = c_k$ and high pass filter $h_1(k) = d_k$. The Fourier transforms of filter h_0 and h_1 are denoted by H_0 and H_1, respectively. The condition for wavelet basis $\Psi(t)$ generated from QMF banks to be compactly supported and orthonormal is provided by the following theorem due to Vaidyanathan [13].

Theorem 2..1 *[13] Let $H_0(z)$ and $H_1(z)$ be causal FIR filters, then the scaling function $\phi(t)$ and wavelet function $\psi(t)$ generated by the QMF bank are causal with finite duration Kb_0. Further, if $H_0(z)$ and $H_1(z)$ satisfy the paraunitary condition, $|H_0(1)| = \sqrt{2}$ and $H_0(e^{j\omega}) \neq 0$ while $|\omega| < \pi/2$, the wavelet functions $\psi_{j,l}(t)$ are orthonormal.*

This theorem imposes constraints on parameter $\{c_k\}$ to generate a compactly supported orthonormal wavelet basis. In particular, the cross-filter orthonormality implied by the paraunitary property, is satisfied by the choice of

$$H_1(z) = -z^{-K} H_0(-z^{-1}), \text{ K odd} \quad (3)$$

or in the time domain,

$$h_1(k) = (-1)^k h_0(K - k). \quad (4)$$

As we can see from the above, both the scaling function and wavelet function depend on the selection of $\{c_k\}$ for $k \in [0, K]$. As a consequence, the dilations and shifts of the mother wavelet depend on the selection of this set of parameters subject to the paraunitary condition imposed on the filters of the QMF bank.

Definition 2..3 *Let H be a Hilbert space which is an orthogonal direct sum*

$$H = \oplus \sum H_i, \quad (5)$$

a map $\mathcal{E}$ is called decomposition entropy if

$$\mathcal{E}(v, \Psi) = -\sum \frac{\|v_i\|^2}{\|v\|^2} \log \frac{\|v_i\|^2}{\|v\|^2} \quad (6)$$

for $v \in H$, $\|v\| \neq 0$, such that

$$v = \oplus \sum v_i, v_i \in H_i, \quad (7)$$

and we set $p \log p = 0$, when $p = 0$.

The implication of using entropy as a performance measure takes advantage of the nonuniform energy distribution of the signal or image in consideration over its energy spectrum. For a source of a finite number of independent signals, such as a digital image considered as a source of independent pixels, its entropy is maximum for uniform distribution [14].

We introduce a cost functional to facilitate the optimization process,

$$\lambda(\Psi, v) = -\sum_j \|v_j\|^2 \log \|v_j\|^2 \quad (8)$$

which relates to the decomposition entropy through

$$\mathcal{E}(v, \Psi) = \|v\|^{-2} \lambda(\Psi, v) + \log \|v\|^2 (2M + 1). \quad (9)$$

The task for constructing an image based optimal wavelet basis becomes one of finding the appropriate filter coefficient $\{c_k\}$ such that the cost functional λ is minimized for the given image. The following theorem provides the analytical gradient of the cost functional (8).

Theorem 2..2 *[11] Let $\lambda(\cdot,\cdot)$ be the additive information measure and $[0, K]$ be the compact support for $\{c_k\}$ and Ψ be the corresponding wavelet basis from dilations and shifts of the wavelet $\psi(t)$. Let $f(t)$ be a fixed signal in $L^2(R)$. Then the gradient of the information measure with respect to the parameter set $\{c_k\}$ for the given signal is described by*

$$\frac{\partial \lambda(\Psi, f(t))}{\partial c_k} = -\sqrt{2^{-j+2}} \sum_j \sum_l \log 2 \|f_j\|^2 \cdot f_{j,l} \sum_n \left[(-1)^{K-k} \left\langle f(t), \phi(2^{-j+1}t - 2l - n)\right\rangle + (-1)^n c_{K-n} \left\langle f(t), \phi(2^{-j+2}t - 4l - 2n - k)\right\rangle\right].$$

This information gradient can be used in computing the filter coefficients for the optimal wavelet basis.

3. IMAGE COMPRESSION

In extending 1-D wavelet to 2-D image applications, we follow Mallat [12] in his hierarchical wavelet decomposition. We then threshold the resulting wavelet coefficients. The retained wavelet coefficients are used to reconstruct the image. In this process, we assume that these coefficients can be transmitted and used precisely.

We have identified the problem of finding an optimal wavelet basis Ψ with that of choosing corresponding parameter set $\{c_k\}$ such that the additive information measure λ is minimized. Next, comes our basis selection algorithm based on the information gradient method [11].

Algorithm 3..1 *Computation of the optimal wavelet basis*

Step 1: Set $i := 1$,
$\lambda_0 := 0$,
Initialize vector C_0;
Input $f(t)$.
Step 2: $C_i := C_{i-1} + p_{i-1} \frac{\partial \lambda}{\partial C_{i-1}}$.
Step 3: *Compute* ϕ *and* ψ.
Step 4: *Compute* λ.
Step 5: *If* $|\lambda_i - \lambda_{i-1}| > \epsilon$,
$i := i + 1$, *go to Step 3.*
Step 6: *Output the optimal basis* Ψ *and stop.*

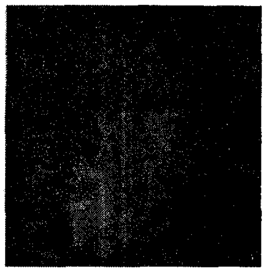

Figure 1. Original 512 by 512 mammographic image.

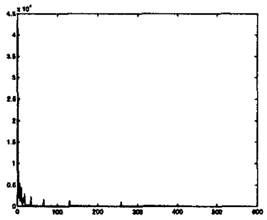

Figure 2. The amplitude of wavelet coefficients of the mammographic image using basis Opt4, listed from low resolution to high resolution components

In the algorithm above, $f(t)$ represents the image data or signals and C denotes the parameter set $\{c_0 c_1 \cdots c_{K-l}\}$. One needs an initial parameter set as a starting point.

We start the optimization scheme based on a low order wavelet system. The smaller the support of the wavelet, the better it can capture the features corresponding to edges. In general, the wavelet decomposition requires less hardware implementation than does the Fourier method. With a lower order system, the cost of implementation shall be further reduced. We first tested compressing the 512 by 512 Lena image by using Daubechies 20, 12 and 4 wavelets. At the same compression ratio, 31.25 : 1, the image represented by the Daub4 basis shows comparable quality when compared against those represented by the two higher order wavelet bases. As a consequence, we select fourth order filters in the optimization process.

4. RESULTS

The optimization is applied to a digital mammographic image shown in Figure 1. This image is obtained through the Department of Radiology, Veterans Administration Medical Center in Baltimore, Maryland. We choose Daubechies' fourth order wavelet coefficients as an initial parameter set to start the optimization procedure with the algorithm above. We denote Daubechies' fourth order wavelet and the optimized wavelet bases by Daub4 and Opt4, respectively. The coefficients of the two corresponding low pass filters are given in Table 1. The amplitude of the wavelet coefficients obtained with wavelet basis Opt4 is illustrated in Figure 2. The coefficients with larger amplitude concentrate on the low resolution region. The histogram in Figure 2 shows the distribution of the wavelet coefficients of the image with basis Opt4.

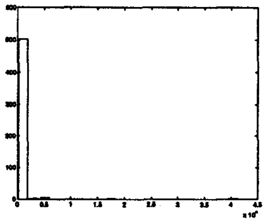

Figure 3. Histogram of the amplitude of wavelet coefficients of the mammographic image.

	Daub4	Opt4
c0	4.8296291e-01	5.2844307e-01
c2	8.3651630e-01	8.0297232e-01
c3	2.2414387e-01	1.8632579e-01
c4	-1.2940952e-01	-1.0352762e-01

Table 1. Daubechies 4 and Opt4 wavelet filter coefficients.

It is obvious that significant compression can be obtained by truncating the large number of small coefficients or by coding them with fewer bits. The threshold is selected by experiments. Different threshold values have been tested to choose one necessary to represent the image without perceptible loss in image quality. The quantitative results in terms of entropy values, peak signal noise ratios and compression ratios are listed in Table 2.

As we can see that a lower entropy value corresponds to a higher compression ratio with a certain loss in PSNR. The reconstructed images using wavelet Daub and Opt4 are illustrated in Figure 4 and Figure 5, respectively. The reconstructed image using basis Opt4 preserves the texture and edges at a level comparable to the one from using basis Daub4. The improvement in the ratio is about ten percent in this case. Although the PSNR is in favor of the Daub4 basis, the actual visual difference is not perceptible.

Similar to other gradient based optimization procedures, this method has its limitations. It often stops at a local minimum and results in a suboptimal solution. However, the suboptimal solution may still provide an acceptable parameter set. The actual wavelet coding system design would include, in addition to finding the optimal basis, using different techniques such as the noise shaping bit allocation procedure [6] or hierarchical coding with the estimated local noise sensitivity of the human vision system(HVS) [15] among others.

5. CONCLUSIONS

This paper has provided a direct approach to construct an image based optimal orthonormal wavelet basis with compact support for image compression. The cost functional, an additive information measure, is introduced based on the

	λ	PSNR	Compression Ratio
Daub4	0.6995	46.3363	18.58:1
Opt4	0.6739	30.7888	20.58:1

Table 2. Entropy values, PSNR and compression ratios from employing Daub4 and Opt4 wavelet bases.

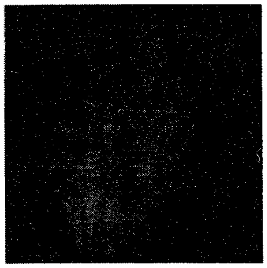

Figure 4. Reconstructed image using Daub4 wavelet, compression ratio 18.58:1.

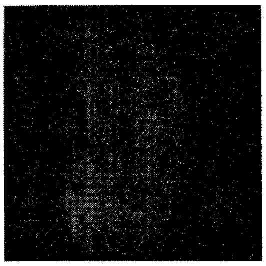

Figure 5. Reconstructed image using Opt4 wavelet, compression ratio 20.53:1.

decomposition entropy of a given image with respect to an initial wavelet basis. Using the resulting optimal wavelet basis improves image compression ratios. The gain in compression outweighs the overhead due to implementing the optimal basis. The parameterization of the cost functionals described in this paper is helpful; other forms of measures or cost functions may be introduced depending on the contexts of actual problems.

This methodology of the optimal basis selection in a general setting is useful not only for image compression, signal approximation and reconstruction, but also for feature analysis, motion estimation in video and HDTV, and system identification. In the context of pattern recognition, it is also a way to construct the feature space.

Future work includes using the optimal wavelet basis for image feature extraction and analysis, and for designing the corresponding bit allocation scheme to maximize the benefits of implementing the signal based wavelet basis.

6. ACKNOWLEDGMENTS

This research was partially supported by NSF grant NSFD CDR 8803012 through the Engineering Research Center's Program.

REFERENCES

[1] C.K. Chui. *Wavelets: A tutorial in Theory and Applications.* Acamemic Press, Inc., 250 Sixty Ave, San Diego, CA 92101, 1992.

[2] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, 3600 University City Science Center, Philadelphia PA, 1992.

[3] O. Rioul and M. Vetterli. Wavelets and signal processing. *IEEE Signal Processing Magazine*, pages 14–38, Oct. 1991.

[4] S.G. Mallat. A theory for multiresolution signal decomposition: The wavelet representation. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 11(7):674–693, July 1989.

[5] S. Mallat and S. Zhong. Characterization of signals from multiscale edges. *IEEE Trans. on Pattern Analysis and Machine intelligence*, 14(7):710–732, July 1992.

[6] M. Antonini *et al.* Image coding using wavelet transform. *IEEE Trans. Image Processing*, pages 205–220, April 1992.

[7] A.H. Tewfik, D. Sinha, and P. Jorgensen. On the optimal choice of a wavelet for signal representation. *IEEE Trans. on Information Theory*, 38(2):747–765, Mar. 1992.

[8] P. Jorgensen. Choosing discrete orthogonal wavelets for signal analysis and approximation. In *IEEE International Conference on Acoustics, Speech and Signal Processing*, pages III–308–311, Minnesota, Minneapolis, April 27-30 1993.

[9] R.R. Coifman and M.V. Wickerhauser. Entropy-based algorithms for best basis selection. *IEEE Trans. on Information Theory*, 38(2):713–718, Mar. 1992.

[10] Y. Zhuang and J.S. Baras. Identification of infinite dimensional systems via adaptive wavelet neural networks. Tech. Report 93-64, Institute for Systems Reseaarch, The University of Maryland, College Park, MD 20742, Aug. 1993.

[11] Y. Zhuang and J.S. Baras. Optimal wavelet basis selectin for signal representation. In Harold H. Szu, editor, *Wavelet Applications*, pages 200–211, Vol. 2242, 1994. SPIE–The International Society for Optical Engineering.

[12] S.G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$. *Transactions of The American Mathematical Society*, 315(1):69–87, Sept. 1989.

[13] P.P. Vaidyanathan. *Multirate Systems and Filterbanks.* Prentice Hall Signal Processing Series. P T R Prentice-Hall, Inc., Englewood Cliffs, NJ 07632, 1993.

[14] A.K. Jain. *Fundamentals of Digital Image Processing.* Prentice-Hall, Inc., Englewood Cliffs, NJ 07632, 1989.

[15] A.S. Lewis and G. Knowles. Image compression using the 2-d wavelet transform. *IEEE Trans. on Image Processing*, 1(2):244–250, Apr. 1992.