Preprints of the 20th ISCIE SSS
Osaka, Nov. 16-18, 1988

Special Lecture

Nonlinear Filtering Asymptotics, Large Deviations and Observers for Nonlinear Systems

John S. Baras and M.R. James
Systems Research Center
University of Maryland
College Park, MD 20742, USA

Abstract

We present a methodology for the construction of dynamic observers for nonlinear systems. The basic methodology used is to construct these observers as limits of nonlinear filters when certain noise parameters tend to zero. We, therefore, make contact with some interesting results on large deviations of filtered diffusion processes. Some examples illustrate the results.

1. Introduction

In this paper we present a design for an observer for the nonlinear control system

$$\begin{aligned} \dot{x} &= f(x,u), \qquad x(0) = x_0, \\ y &= h(x) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $|u_i| \leq 1$ $i = 1, \ldots, m$ and $y \in \mathbb{R}^p$. The initial condition x_0 is unknown.

The *observer problem* consists of recursively computing an estimate $z(t)$ of $x(t)$ for which the error decays to zero as $t \to \infty$. That is, to design a system

$$\begin{aligned} \dot{m} &= F(m,u,y), \qquad m(0) = m_0, \\ z &= G(m) \end{aligned} \tag{1.2}$$

such that

$$\lim_{t \to \infty} |x(t) - z(t)| = 0 \tag{1.3}$$

for all x_0 in a suitable class $\mathcal{I}$. Here $\mathcal{I}$ represents a priori knowledge concerning the initial condition x_0.

Baras and Krishnaprasad [4] have proposed a method for constructing an observer as a limit of nonlinear filters for a family of associated filtering problems, parameterised by $\epsilon > 0$. More recent work in this direction is presented in [1-3]. It is of interest then to study the asymptotic behaviour of the corresponding unnormalised conditional densities $q^\epsilon(x,t)$ as $\epsilon \to 0$, via the Zakai equation (3.2). We obtain the asymptotic formula

$$q^\epsilon(x,t) \;=\; \exp\left(-\frac{1}{\epsilon}\left(W(x,t) + o(1)\right)\right), \tag{1.4}$$

as $\epsilon \to 0$, where $W(x,t)$ is the value function corresponding to a deterministic optimal control problem, namely that arising in deterministic estimation.

Hijab has studied this asymptotic estimation problem, and obtained a WKB expansion when $W(x,t)$ is smooth. This identifies the limiting filter as Mortensen's deterministic or minimum energy estimator. In addition, Hijab has proved a large deviation principle for the conditional measures for the filtering problem (3.1). We extend Hijab's large deviation result by allowing random initial conditions in (3.1), and observe that the resulting variational problem (c.f. action functional) is exactly the optimal control problem mentioned above.

The asymptotic formula for the unnormalised conditional densities (Theorem 3.2) and the large deviation principle for the unnormalised conditional measures (Theorem 3.4) characterise the limiting filter in terms of the deterministic estimator.

We prove the following result for our observer design: provided that we have some knowledge of x_0 (in the form $|x_0 - z_0| < \rho$, where z_0 is the initial estimate) and assuming that (1.1) satisfies a *detectability* condition, then the observer estimate $z(t)$ converges exponentially to the system trajectory $x(t)$ as $t \to \infty$. The radius of convergence ρ depends on the nonlinearities in the dynamics and observations as well as on certain design parameters. For a certain class of systems, no knowledge of x_0 is required.

The results described here are a summary of our recent results in [1-3], where we refer the reader for further details. We remark that these designs do not involve coordinate transformations, canonical forms, local linearization, etc, and seem robust when compared with other designs. However, the designs do involve solving Riccati equations and computing certain matrices and constants.

2. Observer Design

We assume that f, h are smooth with bounded derivatives of orders 1 and 2. Let $N \in L(\mathbb{R}^n, \mathbb{R}^n)$, $R \in L(\mathbb{R}^p, \mathbb{R}^p)$ and assume rank $N = n$ and $R > 0$. Assume $t \mapsto u(t)$ is continuous.

Write $A(x,u) = Df(x,u)$, $H(x) = R^{-1}Dh(x)$, where D denotes gradient in the x variable. Set

$$\|A\| = \sup\{\|A(x,u)\| : \; x \in \mathbb{R}^n, \; |u_i| \le 1\} \tag{2.1}$$

and similarly define $\|H\|$, and so on.

Consider the coupled system

$$\begin{aligned} \dot{m}(t) &= f(m(t),u(t)) + Q(t)H(m(t))'R^{-1}(y(t) - h(m(t))(2.1) \\ m(0) &= m_0 \\ \dot{Q}(t) &= A(m(t),u(t))Q(t) + Q(t)A(m(t),u(t))' \\ &\quad - Q(t)H(m(t))'H(m(t))Q(t) + NN' \\ Q(0) &= Q_0 > 0. \end{aligned} \tag{2.2}$$

This is our observer for (1.1). It is essentially a modification of the deterministic or minimum energy estimator, as discussed in Baras, Bensoussan and James [1]. Note in particular that the Riccati differential equation (2.2) depends on the control.

Let $P_0 = Q_0^{-1}$, $P(t) = Q(t)^{-1}$, and let p, q be the bounds for $\|P(t)\|$, $\|Q(t)\|$ (given in [3]), and regard A_0, N, R as design parameters. Define $\rho = \rho(Q_0, N, R)$ by

$$\rho = \frac{r_0}{q^2\|P_0^{1/2}\|}\left(\sqrt{p}\|D^2 f\| + \sqrt{q}\|R^{-1}\|^2 \, \|Dh\| \, \|D^2 h\|\right)^{-1} \tag{2.3}$$

Then we have

Theorem 2.1 [3] *Assume there exist Q_0, N, R such that*

$$|x_0 - m_0| < \rho(Q_0, N, R) \tag{2.4}$$

Then the system (2.1), (2.2) *is an observer for the nonlinear control system* (1.1) *provided that* $(H(x), A(x,u))$ *is uniformly detectable and the above assumptions hold. That is, there exist contants $K > 0, \gamma > 0$ such that*

$$|x(t) - m(t)| \le K|x_0 - m_0|e^{-\gamma t} \tag{2.5}$$

for all $t \ge 0$.

By using different estimates for the nonlinearities, we obtain an observer for (1.1) without any contraints on the initial conditions x_0, m_0 for a class of systems. Included in this class are systems for which $A(x,u)$ is uniformly negative definite.

Define $\delta = \delta(Q_0, N, R)$ by

$$\delta = \frac{r_0}{q^2} - 4p\|Df\| - 4\|R^{-1}\|^2 \|Dh\|^2. \tag{2.6}$$

Corollary 2.1 [3] *Assume there exist Q_0, N, R such that*

$$0 < \delta(Q_0, N, R). \tag{2.7}$$

Then the system (2.1), (2.2) is an observer for the control system (1.1) provided that $(H(x), A(x,u))$ is uniformly detectable and the above assumptions hold. That is, there exist contants $K > 0, \gamma > 0$ such that

$$|x(t) - m(t)| \leq K|x_0 - m_0|e^{-\gamma t} \tag{2.8}$$

for all $t \geq 0$ and all $x_0, m_0 \in \mathbb{R}^n$.

3. Nonlinear Filtering Asymptotics and Large Deviations

We consider a family of diffusion processes in $\mathbb{R}^n$ with real valued observations:

$$\begin{aligned} dx^\epsilon(t) &= f(x^\epsilon(t))dt + \epsilon^{1/2}dw(t), \quad x^\epsilon(0) = x_0^\epsilon, \\ dy^\epsilon(t) &= h(x^\epsilon(t))dt + \epsilon^{1/2}dv(t), \quad y^\epsilon(0) = 0. \end{aligned} \tag{3.1}$$

The *Zakai equation* for an unnormalised conditional density $q^\epsilon(x,t)$ is

$$\begin{aligned} dq^\epsilon(x,t) &= A_\epsilon^* q^\epsilon(x,t)dt + \frac{1}{\epsilon}h(x)q^\epsilon(x,t)dy^\epsilon(t), \\ q^\epsilon(x,0) &= q_0^\epsilon(x). \end{aligned} \tag{3.2}$$

Defining

$$p^\epsilon(x,t) = \exp\left(-\frac{1}{\epsilon}y^\epsilon(t)h(x)\right) q^\epsilon(x,t), \tag{3.3}$$

the *robust* form of the Zakai equation is obtained.

Following Fleming and Mitter, we apply the logarithmic transformation

$$S^\epsilon(x,t) = -\epsilon \log p^\epsilon(x,t). \tag{3.4}$$

Then $S^\epsilon(x,t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}S^\epsilon(x,t) - \frac{\epsilon}{2}\Delta S^\epsilon(x,t) + H^\epsilon(x,t,DS^\epsilon(x,t)) &= 0, \\ S^\epsilon(x,0) &= S_0(x) - \epsilon \log C_\epsilon, \end{aligned} \tag{3.5}$$

where

$$H^\epsilon(x,t,\lambda) = \lambda g(x,t) + \frac{1}{2} |\lambda|^2 - V^\epsilon(x,t). \tag{3.6}$$

Equation (3.5) is a nonlinear parabolic PDE, which can be interpreted as the Bellman equation for a stochastic control problem.

Formally letting $\epsilon \to 0$ we obtain a Hamilton–Jacobi equation [2]

$$\frac{\partial}{\partial t}S(x,t) + H(x,t,DS(x,t)) = 0, \quad S(x,0) = S_0(x), \tag{3.7}$$

We shall interpret solutions of (3.7) in the viscosity sense. If we define

$$W(x,t) = S(x,t) - y(t)h(x), \quad y \in \Omega_0, \tag{3.8}$$

then, for $y \in \Omega_0 \cap C^1$, $W(x,t)$ satisfies a Hamilton–Jacobi equation, which is the Bellman equation for the deterministic estimation control problem.

Given an observation record $\mathcal{Y}_t = \{y(s),\ 0 \le s \le t\},\ 0 \le t \le T$, of the deterministic system

$$\begin{aligned} \dot{x} = f(x) \ + \ u, \ \ x(0) \ &= \ x_0, \\ \dot{y} = h(x) \ + \ v, \ \ y(0) \ &= \ 0, \end{aligned} \tag{3.9}$$

we wish to estimate the state at time t, the initial condition x_0 being unknown. Define

$$\tilde{J}_t(x_0, u, v) \ = \ S_0(x_0) + \frac{1}{2}\int_0^t (\mid u(s) \mid^2 + v(s)^2)\, ds. \tag{3.10}$$

A minimum energy input triple (x_0^*, u^*, v^*) given $\mathcal{Y}_t$ is a triple that minimises $\tilde{J}_t$ subject to the constraint that the trajectory of (3.9) produces the output $\mathcal{Y}_t$. By replacing $v(s)$ by $\dot{y}(s) - h(x(s))$ in (3.10) and omitting the $\dot{y}(s)^2$ term, we can formulate an equivalent unconstrained optimal control problem. Define

$$J_t(x_0, u) \ = \ S_0(x_0) + \int_0^t L(x(s), u(s), s) ds, \tag{3.11}$$

where

$$L(x, u, s) \ = \frac{1}{2} \mid u \mid^2 + \frac{1}{2} h(x)^2 - \dot{y}(s) h(x). \tag{3.12}$$

We now minimise J_t over pairs (x_0, u). The *deterministic* or minimum energy *estimate* $\hat{x}(t)$ given $\mathcal{Y}_t$ is defined to be the endpoint of the optimal trajectory $s \mapsto x^*(s),\ 0 \le s \le t$, corresponding to a minimum energy pair (x_0^*, u^*) : $\hat{x}(t) = x^*(t)$.

Define a *value function*

$$W(x, t) \ = \ \inf_{(x_0, u) \ \in \ \mathcal{U}_{x,t}} J_t(x_0, u). \tag{3.13}$$

By using standard methods, we see that $W(x,t)$ is continuous and formally satisfies the *Bellman equation*

$$\frac{\partial}{\partial t} W(x,t) \ + \ \tilde{H}(x, t, DW(x,t)) = 0, \ \ W(x, 0) = S_0(x), \tag{3.14}$$

To obtain $\hat{x}(t)$, one minimises $W(x,t)$ over x.

Theorem 3.1 [2] *The value function $W(x,t)$ defined by (3.13) is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation (3.14).*

We can now state

Theorem 3.2 [2] *Under the above assumptions, we have*

$$\lim_{\epsilon \to 0} \epsilon \log q^\epsilon(x, t) \ = \ -W(x, t) \tag{3.15}$$

uniformly on compact subsets of $\mathbb{R}^n \times [0, T]$, where $W(x,t)$ is defined by (3.8).

We have seen that the optimal control problem associated with deterministic estimation plays a key role in studying the asymptotics of the Zakai equation. This control problem is exactly the variational problem arising in a large deviation principle for certain conditional measures.

Fix x_0 and consider the stochastic differential equation (3.1) with initial condition $x_0^\epsilon = x_0$ for all $\epsilon > 0$. Let $Q^\epsilon_{x|(y,x_0)}$ be an unnormalised conditional measure on $\Omega^n = C([0,T], \mathbb{R}^n)$ of x^ϵ given $y \in \Omega_0$ and the initial condition x_0. Given a control $t \to u(t)$, let x_u denote the corresponding trajectory of (3.9). Hijab proved the following.

Theorem 3.3 *For any open subset $\mathcal{O}$ and any closed subset $\mathcal{C}$ of Ω^n,*

$$\liminf_{\epsilon \to 0} \epsilon \log Q^\epsilon_{x|(y,x_0)}(\mathcal{O}) \ge -I(x_0, y, \mathcal{O})$$
$$\limsup_{\epsilon \to 0} \epsilon \log Q^\epsilon_{x|(y,x_0)}(\mathcal{C}) \le -I(x_0, y, \mathcal{C})$$

where for $\mathcal{A} \subset \Omega^n$,

$$I(x_0, y, \mathcal{A}) = \inf_u \left\{ \frac{1}{2} \int_0^T (| u(s) |^2 + h(x_u(s))^2) \, ds \right. \tag{3.16}$$
$$\left. - \int_0^T h(x_u(s)) dy(s) \mid x_u(0) = x_0, \; x_u \in \mathcal{A} \right\},$$

with the understanding that the infimum over an empty set is infinite.

Now let the initial conditions of (3.1) be random with unnormalised density defined by (3.2). Let $Q^\epsilon_{(x,x_0)|y}$ be an unnormalised joint conditional measure of $(x^\epsilon, x_0^\epsilon)$ on $\Omega^n \times \mathbb{R}^n$ given $y \in \Omega_0$.

Theorem 3.4 [2] *For any open subset $\mathcal{O}$ and any closed subset $\mathcal{C}$ of Ω^n, and for any open subset $\mathcal{O}_0$ and any closed subset $\mathcal{C}_0$ of $\mathbb{R}^n$, we have*

$$\liminf_{\epsilon \to 0} \epsilon \log Q^\epsilon_{(x,x_0)|y} (\mathcal{O} \times \mathcal{O}_0) \geq -J(\mathcal{O} \times \mathcal{O}_0, y) \tag{3.17}$$

$$\limsup_{\epsilon \to 0} \epsilon \log Q^\epsilon_{(x,x_0)|y} (\mathcal{C} \times \mathcal{C}_0) \leq -J(\mathcal{C} \times \mathcal{C}_0, y) \tag{3.18}$$

where for $\mathcal{A} \times \mathcal{A}_0 \subset \Omega^n \times \mathbb{R}^n$,

$$J(\mathcal{A} \times \mathcal{A}_0, y) \; = \; \inf_{x_0 \in \mathcal{A}_0} \{ S_0(x_0) \; + \; I(x_0, y, \mathcal{A}) \}. \tag{3.19}$$

Note that the variational problem (3.19) corresponds to the optimal control problem (3.9) - (3.13). Theorem 3.4 implies that the limiting measure is concentrated on the optimal initial condition x_0^* and optimal trajectory $x^*(s)$, $0 \leq s \leq T$.

4. Examples

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \tag{4.1}$$
$$y = \sin x_1.$$

This system is controllable and observable. However, the pair of matrices $(Dh(x), A)$ is not observable for $x_1 = k\frac{\pi}{2}$, where k is an odd integer. The system has eigenvalues $-1, -2$ and A is symmetric, hence $(Dh(x), A)$ is automatically uniformly detectable, with $\alpha_0 = 1, \Lambda(x) \equiv 0$. Let $R = rI$, $N = \sqrt{r_0}I$, $Q_0 = \gamma^2 I$. Here, $H(x) = \frac{1}{r}(\cos x_1, 0)$. Now

$$\delta = r_0(\gamma^2 + r_0/2)^{-2} - 4r^{-2}.$$

Set $r = 3$, $r_0 = 0.2$, $\gamma = 0.1$. Then $\delta = 7.82$.
The observer for (4.1) is

$$\dot{m}(t) = Am(t) + Bu(t) + \frac{1}{3} Q(t) \, H(m(t))' (y(t) - \sin m_1(t)), \tag{4.2}$$
$$\dot{Q}(t) = AQ(t) + Q(t)A' - Q(t)H(m(t))'H(m(t))Q(t) + 0.2I.$$

By Corollary 2.1, $m(t)$ converges exponentially to $x(t)$ for all $x_0, m_0 \in \mathbb{R}^n$.

References

[1] J.S. Baras, A. Bensoussan and M.R. James, *Dynamic Observers as Asymptotic Limits of Recursive Filters: Special Cases, SIAM J. Applied Math.*, Vol. 48, No. 5, October 1988 pp.1147-1158.

[2] M.R. James and J.S. Baras, *Nonlinear Filtering and Large Deviations: A PDE-Control Theoretic Approach*, Stochastics, Vol. 23, pp.391-412, 1988.

[3] M.R. James and J.S. Baras, *An Observer Design for Nonlinear Control Systems*, Proceedings of the 1988 INRIA Conference on Optimization of Systems, Antibes, France, 1988.

[4] J.S. Baras and P.S. Krishnaprasad, *Dynamic Observers as Asymptotic Limits of Recursive Filters*, Proc. of IEEE 21st CDC, 1982, pp.1126-1127.