GROUP INVARIANCE METHODS IN NONLINEAR FILTERING OF DIFFUSION PROCESSES

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ABSTRACT

Given two "nonlinear filtering problems" described by the processes

$$dx^i(t) = f^i(x^i(t))\,dt + g^i(x^i(t))\,dw^i(t)$$
$$dy^i(t) = h^i(x^i(t))\,dt + dv^i(t)\ , \quad i=1,2, \tag{1}$$

we define a notion of strong equivalence relating the solutions to the corresponding Mortensen-Zakai equations

$$du_i(t,x) = \overset{i}{\mathcal{L}}u_i(t,x)dt + \mathcal{L}_1^i u_i(t,x)dy_t^i\ , \quad i=1,2, \tag{2}$$

which allows solution of one problem to be obtained easily from solutions of the other. We give a geometric picture of this equivalence as a group of local transformations acting on manifolds of solutions. We then show that by knowing the full invariance group of the time invariant equations

$$du_i(t,x) = \overset{i}{\mathcal{L}}u_i(t,x)dt\ , \quad i=1,2, \tag{3}$$

we can analyze strong equivalence for the filtering problems. In particular if the two time invariant parabolic operators are in the same orbit of the invariance group we can show strong equivalence for the filtering problems. As a result filtering problems are separated into equivalent classes which correspond to orbits of invariance groups of parabolic operators. As specific example we treat V. Beneš's case establishing from this point of view the necessity of the Riccati equation.

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1. INTRODUCTION

Very recently new ideas and techniques have been applied to a long standing problem in stochastic systems theory: "the nonlinear filtering problem". A large portion of this new work is geometrical in nature. Thus Brockett [1]-[2] and Mitter [3]-[4] have emphasized the significance of certain Lie-algebra of partial differential operators associated with each nonlinear filtering problem, while Marcus et al [5] and Baras and Blankenship [6] have provided explicit examples where these concepts lead to significant developments in the solution of nonlinear filtering problems.

Our objective here is to describe a geometric way of characterizing computationally equivalent nonlinear filtering problems. This work is inspired by similar ideas in the theory of ordinary differential equations which go under the names "Similarity Methods" or "Group Invariance Methods"[9]-[10]. It will be apparent from the present paper that the fundamental concept in this problem of "equivalence" is that of invariance groups of (4). To make things precise consider two nonlinear filtering problems (vector) as in (1) of the abstract and the corresponding Mortensen-Zakai equations in Stratonovich form

$$\frac{\partial u_i(t,x)}{\partial t} = (\mathcal{L}^i - \frac{1}{2}\| h^i(x)\|^2)u_i(t,x) + h^{iT}(x)u_i(t,x)y^i(t), \quad (4)$$
$$i=1,2$$

<u>Definition</u>: The two nonlinear filtering problems above are <u>strongly equivalent</u> if u_2 can be computed from u_1, and vice versa, via the following types of operations:

<u>Type 1</u>: $(t,x^2) = \alpha(t,x^1)$, where α is a diffeomorphism.

<u>Type 2</u>: $u_2(t,x) = \psi(t,x)u_1(t,x)$, where $\psi(t,x)\geq 0$ and $\psi^{-1}(t,x)\geq 0$.

<u>Type 3</u>: Solving a set of ordinary (finite dimensional) differential equations (i.e. quadrature).

Brockett [2], has analyzed the effects of diffeomorphisms in x-space and he and Mitter [4] the effects of so called "gauge" transformations (a special case of our type 2 operations) on (4). Type 3 operations are introduced here for the first time, and will be seen to be the key in linking this problem with mathematical work on group invariance methods in o.d.e. and p.d.e.'s.

Our approach starts from the abstract version of (4):

$$\frac{\partial u_i}{\partial t} = (A^i + \sum_{j=1}^{p} B_j^i \dot{y}_j(t))u_i; \quad i=1,2 , \quad (5)$$

where A^i, B_j^i are given by

$$\begin{aligned} &\mathcal{L}^i := \text{forward gen. of } x^i \\ &A^i := \mathcal{L}^i - \tfrac{1}{2} h^{iT} h^i \\ &B^i_j := \text{Mult. by } h^i_j \text{ (jth comp. of } h^i) . \end{aligned} \tag{6}$$

We are thus dealing with two parabolic equations. We will first examine whether the evolutions of the time invariant parts can be computed from one another. This is a classical problem and the methods of section 2 apply. In section 3 we shall give an extension to the full equation (5) under certain conditions on B^i_j. We shall then apply this result to the examples studied by Beneš and recover the Riccati equation as a consequence of strong equivalence. Further results and details can be found in [8].

The estimation Lie algebra introduced by Brockett [2] is the Lie algebra.

$$\Lambda^i(E) = \text{Lie algebra generated by } A^i \text{ and } B^i_j \text{ , } j=1,\ldots p, \quad i=1,2 . \tag{7}$$

Again we shall assume that for problems considered the operators A, B_j have a common, dense invariant set of analytic vectors in X and that the mathematical relationship between $\Lambda^i(E)$ and the existence-uniqueness theory of (4) is well understood. For results of this nature we refer to [6][7].

2. USING THE INVARIANCE GROUP OF A PARABOLIC P.D.E. IN SOLVING NEW P.D.E.'S

Consider the general, linear, nondegenerate elliptic partial differential operator

$$L := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)\,\text{id}. \tag{8}$$

and assume that the coefficients a_{ij}, b_i, c are smooth enough, so that L generates an analytic semigroup, denoted by exp(tL), for at least small $t \geq 0$, on some locally convex space X of initial functions ϕ and appropriate domain Dom(L).

Let V be the set of solutions to

$$\begin{aligned} &\frac{\partial u}{\partial t} = Lu \\ &u(0,x) = \phi(x) \end{aligned} \tag{9}$$

in X, as we vary ϕ. The aim is to find a local Lie transformation group G which transforms every element of V into another element of V. Such a group will be called an <u>invariance group</u> of (9) or of L. This of course is a classical topic of mathematical research initiated by Sophus Lie.

Theorem 1 [10]: Every transformation g in the invariance group G of a linear parabolic equation is of the form

$$u(t,x) \to \nu(p(t,x))u(p(t,x)) + \psi(x) \tag{10}$$

where p is a transformation acting on the variables (t,x), ψ a fixed solution of the parabolic equation.

We consider now one-parameter subgroups of the invariance group G of a given parabolic partial differential equation. According to standard Lie theory the infinitesimal generators of these one-parameter subgroups form the Lie algebra $\Lambda(G)$ of the local Lie group G [9]. We shall, using standard Lie theory notation, denote X_s by exp(sX) where X is the infinitesimal generator of the one parameter group $\{X_s\}$. The infinitesimal generators of G are given by

$$Z = \alpha(t,x)\frac{\partial}{\partial t} + \sum_{i=1}^{n}\beta_i(t,x)\frac{\partial}{\partial x_i} + \gamma(t,x)\,\mathrm{id}. \tag{11}$$

for some functions α,β_i,γ of t and x. If u solves (9) so does v(s)=exp(sZ)u, for small s. However v is also the solution of

$$\frac{\partial v}{\partial s} = \alpha\frac{\partial v}{\partial t} + \sum_{i=1}^{n}\beta_i\frac{\partial v}{\partial x_i} + \gamma v\ , \quad v(0) = u, \tag{12}$$

a first order hyperbolic p.d.e. (solvable by the method of characteristics). Clearly since $\partial/\partial t - L$ is linear

$$Zu \in V \ \text{ if } u \in V. \tag{13}$$

(13) implies

$$dZ/dt = [L,Z] \text{ on } V\ . \tag{14}$$

In (14) [,] denotes commutator and dZ/dt is symbolic of $\alpha_t\frac{\partial}{\partial t} + \sum_{i=1}^{n}\beta_{i,t}\frac{\partial}{\partial x_i} + \gamma_t$ id. Thus the elements of $\Lambda(G)$ in this case satisfy a Lax equation. Furthermore it can be shown [10] that α is independent of x, i.e. $\alpha(t,x) = \alpha(t)$ and that every Z satisfies an o.d.e.

$$d_\ell\frac{d^\ell Z}{dt^\ell} + d_{\ell-1}\frac{d^{\ell-1}Z}{dt^{\ell-1}} + \ldots + d_0 Z = 0$$

where $\ell \le \dim G$.

The most widely known example, for which $\Lambda(G)$ has been computed explicitly is the heat equation. The infinitesimal generators in this case are six, as below

$$\frac{\partial}{\partial t}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \ 1, 2t\frac{\partial}{\partial x} + x, \ 4t^2 \frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + x^2 \ . \tag{15}$$

Knowing the invariance group of (9) can help in solving certain "perturbations" of (9). We folow Rosencrans [10]. Thus we consider a linear parabolic equation like (9) and we assume we know the infinitesimal generators Z of the nontrivial part of G. Thus if u solves (9), so does $v(s)=\exp(sZ)u$ but with some new initial data, say, $R(s)\phi$. That is

$$e^{sZ}e^{tL} = e^{tL}R(s) \text{ on } X. \tag{16}$$

Now $R(\cdot)$ is a semigroup. Let M be its generator:

$$M\phi = \lim_{s\to 0} \frac{R(s)\phi-\phi}{s} \ , \ \phi\varepsilon\mathrm{Dom}(M) \ . \tag{17}$$

It is straightforward to compute M, given Z as in (16). Thus

$$M\phi = \alpha(0)L\phi + \sum_{i=1}^{n}\beta_i(0,x)\frac{\partial\phi}{\partial x_i} + \gamma(0,x)\phi \ . \tag{18}$$

Thus, to solve the initial value problem

$$\partial w/\partial s = Mw \ , \ w(0) = \phi \tag{19}$$

where

$$M = \alpha(0)L + \sum_{i=1}^{n}\beta_i(0,x)\frac{\partial}{\partial x_i} + \gamma(0,x) \text{ id}$$

we follow the steps given below.

Step 1: Solve $u_t = Lu$, $u(0) = \phi$.

Step 2: Find generator Z of G corresponding to M and solve

$$\frac{\partial v}{\partial s} = \alpha(t)\frac{\partial v}{\partial t} + \sum_{i=1}^{n}\beta_i(t,x)\frac{\partial v}{\partial x_i} + \gamma(t,x)v \ , \ v(0) = u \tag{20}$$

via the method of characteristics. Note this step requires the solution of ordinary differential equations only.

Step 3: Set $t=0$ to $v(s,t,x)$.

This procedure allows easy computation of the solution to the "perturbed" problem (19) if we know the solution to the "unperturbed" problem (9). The "perturbation" which is of degree $\leq$ 1st, is given by the part of M:

$$P = \sum_{i=1}^{n} \beta_i(0,x)\frac{\partial}{\partial x_i} + \gamma(0,x).\ \text{id}. \tag{21}$$

We shall denote by $\Lambda(P)$ the set of all perturbations like (21).

<u>Definition</u>: The Lie algebra $\Lambda(P)$ will be called the <u>perturbation algebra</u> of the elliptic operator L.

<u>Theorem 2 [10]</u>: The perturbation algebra $\Lambda(P)$ of an elliptic operator L, is isomorphic to a Lie subalgebra of $\Lambda(G)$(i.e. of the Lie algebra of the invariance group of L). Moreover $\dim(\Lambda(P)) = \dim(\Lambda(G))-1$.

One significant question is: can we find the perturbation algebra $\Lambda(P)$ without first computing $\Lambda(G)$, the invariance Lie algebra? The answer is affirmative and is given by the following result [10].

<u>Theorem 3 [10]</u>: Assume L has analytic coefficients. An operator P_0 of order one or less (i.e. of the form (21) is in the perturbation algebra $\Lambda(P)$ of L iff there exist a sequence of scalars $\lambda_1,\lambda_2,\ldots$ and a sequence of operators $P_1,P_2,\ldots$ of order less than or equal to one such that $[L,P_n]=\lambda_n L+P_{n+1}$, $n\geq 0$ and $\Sigma\lambda_k t^k/k!$, $\Sigma P_k t^k/k!$ converge at least for small t.

It is an easy application of this result to compute the perturbation algebra of the heat equation in one dimension or equivalently of $L = \partial^2/\partial x^2$. It turns out that $\Lambda(P)$ is a 5-dimensional and spanned by

$$\Lambda(P) = \text{Span}(1,x,x^2, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}). \tag{22}$$

So the general perturbation for the heat equation looks like

$$P = (ax+b)\frac{\partial}{\partial x} + (cx^2 + dx + e)\,\text{id} \tag{23}$$

where a,b,c,d,e are arbitrary constants.

The implications of these results are rather significant. Indeed consider the class of linear parabolic equations $u_t=Lu$, where L is of the form (8). We can define an equivalence relationship on this class by : "L_1 is <u>equivalent</u> to L_2 if $L_2=L_1+P$ where P is an element of the perturbation algebra $\Lambda^1(P)$ of L_1". Thus elliptic operators or equivalently linear parabolic equations are divided into equivalent classes (orbits); within each class (orbit) $\{L(k)\}$ (k indexes elements in the class) solutions to the initial value problem $u(k)_t=L(k)u(k)$ with fixed data ϕ (independent of k) can be obtained by quadrature (i.e. an o.d.e. integration) from any one solution $u(k_0)$.

3. SUFFICIENT CONDITIONS FOR STRONG EQUIVALENCE AND APPLICATIONS.

We return now to the problem posed in section 1. Namely to discover conditions that imply strong equivalence of two nonlinear filtering problems. Our main result is:

Theorem 4: Given two nonlinear filtering problems (see (1)(2)), such that the corresponding Mortensen-Zakai equations (see (4)) have unique solutions, continuously dependent on y(·). Assume that using operations of type 1 and 2 (see definition in section 1) these stochastic p.d.e. can be transformed in bilinear form

$$\frac{\partial u_i}{\partial t} = (A^i + \sum_{j=1}^{p} B_j^i \xi_j^i(t))u_i, \quad i=1,2$$

such that: (i) $A^i, i=1,2$, are nondegenerate elliptic, belonging to the same equivalence class (see end of section 2) (ii) $B_j^i, j=1,\dots p$, $i=1,2$ belong to the perturbation algebra $\Lambda(P)$ of (i). Then the two filtering problems are strongly equivalent.

Proof: See [8].

Let us apply this result to the Beneš case[11]. We consider the linear filtering problem (scalar x,y)

$$dx(t)=dw(t) \ , \ dy(t)=x(t)dt+dv(t) \tag{24}$$

and the nonlinear filtering problem (scalar x,y)

$$dx(t)=f(x(t))dt+dw(t) \ , \ dy(t)=x(t)dt+dv(t) \ . \tag{25}$$

The corresponding Mortensen-Zakai equations in Stratonovich form are: for the linear

$$\frac{\partial u_1(t,x)}{\partial t} = \frac{1}{2}(\frac{\partial^2}{\partial x^2} - x^2)u_1(t,x)+x\dot{y}(t)u_1(t,x) \ ; \tag{26}$$

for the nonlinear

$$\frac{\partial u_2}{\partial t} = \frac{1}{2}(\frac{\partial^2}{\partial x^2} - x^2)u_2(t,x) - \frac{\partial}{\partial x}(fu_2)+x\dot{y}(t)u_2(t,x). \tag{27}$$

We wish to show that (24)(25) are strongly equivalent only if f (the drift) is a global solution of the Riccati equation

$$f_x + f^2 = ax^2 + bx + c \ . \tag{28}$$

First let us apply to (25)(27) an operation of type 2. That is let (defines v_2)

$$u_2(t,x) = v_2(t,x)\exp\left(\int_0^x f(u)du\right). \tag{29}$$

Then the new function v_2 satisfies

$$\frac{\partial v_2(t,x)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2 - V(x)\right)v_2(t,x) + x\dot{y}(t)v_2(t,x) \ , \tag{30}$$

where

$$V(x) = f_x + f^2 \ . \tag{31}$$

We apply the theorem to (26)(30). So

$$A^1 = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right) \ , \ A^2 = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2 - V\right)$$

while $B^1 = B^2$ = Mult. by x . From the results of section 2, the only possible equivalence class is that of the heat equation. Clearly from (23) $A^1, B^1, B^2 \varepsilon \Lambda(P)$ for this class. For $A^2 \varepsilon \Lambda(P)$ it is necessary that V be quadratic, i.e. f satisfies the Riccati equation (28).

REFERENCES

[1] R.W. Brockett and J.M.C. Clark, "Geometry of the Conditional Density Equation", Proc.Int.Conf. on An. and Opt. of Stoch. Syst., Oxford, England, 1978.

[2] R.W. Brockett, "Classification and Equivalence in Estimation Theory", Proc.18th IEEE Conf.on Dec. and Control, 1979, pp.172-175.

[3] S.K. Mitter, "Filtering Theory and Quantum Fields" to appear in Asterisque, 1980.

[4] S.K. Mitter, "On the Analogy Between Mathematical Problems of Non-linear Filtering and Quantum Physics", Rich.di Automatica, 1980.

[5] S.I. Marcus, M. Hazewinkel and Chugn' huan Lie, "Algebraic Structures in Nonlinear Estimation" Proc. of 1980 Joint Aut.Cont.Conf.

[6] J.S. Baras and G.L. Blankenship, "Nonlinear Filtering of Diffusion Processes: A Generic Example" to appear.

[7] J.S. Baras, S.K. Mitter and D. Ocone, "Nonlinear Filtering of Diffusion Processes in Unbounded Domains", to appear.

[8] J.S. Baras, "Group Invariance Methods in Nonlinear Filtering of Diffusion Processes", Proc. of 1980 IEEE Conf. on Dec. and Cont.

[9] G.W. Bluman and J.C. Cole, Similarity Methods for Differential Equations, Springer-Verlag, 1974.

[10] S.I. Rosencrans, "Perturbation Algebra of an Elliptic Operator", J. of Math. Anal. and Appl., 56, 1976, pp. 317-329.

[11] V. Beneš, "Exact Finite Dimensional Filters for Certain Diffusions with Nonlinear Drift", to appear in Stochastics, 1980.

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