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SOME CONTROLLABILITY PROPERTIES OF BILINEAR DELAY-DIFFERENTIAL SYSTEMS

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Abstract

Bilinear delay-differential systems appear frequently in applications. We present here controllability properties of such systems and in particular we derive criteria for local accessibility, and a "bang-bang" theorem.

Summary

The analysis of finite dimensional bilinear systems [1-7] and of linear delay-differential systems [8-11] has reached a very satisfactory level to date. The purpose of this paper is to present several controllability properties of bilinear delay-differential systems. Abundant examples of this type of systems come from integrated circuits, nuclear reactors and economics [12]. For simplicity we chose to consider only systems of the simple type

$$\frac{dx(t)}{dt} = (A+\sum_{i=1}^{p} u_i(t)B_i)x(t)+ Cx(t-\tau) \qquad (1)$$

where $x(t)\in\mathbb{R}^n$, $u_i(\cdot)$ are scalar functions measurable and bounded on finite intervals and A, B_i, C are nxn matrices. The dynamical characteristics of bilinear delay-differential systems can be found in [12,14]. The natural state space for (1) is a subset of $C([-\tau,0];\ \mathbb{R}^n)$, and we let

$$x_t(\theta)=x(t+\theta);\ \theta\in[-\tau,0] \qquad (2)$$

be the state of (1) at time t, with x(t) a euclidean trajectory of (1). The reachable set in $\mathbb{R}^n$ from initial condition φ, at time $t>0$ will be denoted by $R(t,\varphi)$, and it is the set of all $y\in\mathbb{R}^n$ such that $x(t;0,\varphi,u)=y$ for some admissible control u. The reachable set in $\mathbb{R}^n$ from initial condition φ, in time $t>0$ is $\mathbb{R}(t,\varphi)=\bigcup_{s\le t} R(t,\varphi)$. The reachable set in $\mathbb{R}^n$ from initial condition φ is $\mathbb{R}(\varphi)=\bigcup_{t\ge 0}\mathbb{R}(t,\varphi)$. Let C denote $C\{[-\tau,0];\mathbb{R}^n\}$ and C^1 denote $C^1\{[-\tau,0];\mathbb{R}^n\}$. Then the reachable set in C^1 from initial condition φ, at time $t>0$, will be denoted by $R_C(t,\varphi)$, and it is the set of all $\lambda\in C^1$ s.t. $\lambda(\theta)=x_t(\theta)$, $\theta\in[-\tau,0]$ for some admissible control u. Similarly the reachable set in C^1 from initial condition φ, in time $t>0$, is $\mathbb{R}_C(t,\varphi)=\bigcup_{0\le s\le t} R_C(t,\varphi)$ and the reachable set in C^1 from initial condition φ, is $\mathbb{R}_C(\varphi)=\bigcup_{t\ge 0}\mathbb{R}_C(t,\varphi)$.

We have now the following definitions:

Definition 1: Let $\lambda(\cdot)=x(\cdot;0,\varphi,u)$ be a trajectory of the system. The system has the local accessibility property along λ, in $\mathbb{R}^n$, at time t_1 if there exists an $\mathbb{R}^n$-neighborhood of $x(t_1;0,\varphi,u)$ which is included in $R(t_1,\varphi)$.

Definition 2: The system is euclidean controllable (resp. at time t_1, in time t_1) from initial condition φ if $\mathbb{R}(\varphi)=\mathbb{R}^n$ (resp $R(t_1,\varphi)=\mathbb{R}^n$, $\mathbb{R}(t_1,\varphi)=\mathbb{R}^n$.)

Definition 3: The system is completely euclidean controllable (at time t_1, in time t_1) if it is euclidean controllable (at time t_1, in time t_1) from every initial condition φ.

Consider the general nonlinear differential delay system

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t)) \qquad (3)$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^p$, f is continuously differentiable in all arguments and $f(t,0,0,0)=0$, and the linearized system about the trajectory $x_0(t)=x(t;0,y,u_0)$:

$$\dot{y}(t)=A(t)y(t)+C(t)y(t-\tau)+B(t)u(t) \qquad (4)$$

where

$$A(t)=\frac{\partial}{\partial x} f(t,x(t),x(t-\tau),u(t))]_{x_0,u_0}$$

$$C(t)=\frac{\partial}{\partial x_{-\tau}} f(t,x(t),x(t-\tau),u(t))]_{x_0,u_0}$$

$$B(t)=\frac{\partial}{\partial u} f(t,x(t),x(t-\tau),u(t))]_{x_0,u_0}$$

where u_0 is an admissible control and $x_{-\tau}(t)=x(t-\tau)$.

Theorem 1: Suppose that system (4) is completely euclidean controllable at time t_1. Then the nonlinear delay-differential system (3) has the local euclidean accessibility property along x_0 at t_1.

The proof can be found in [12,14]. Certainly Theorem 1 applies to bilinear delay differential systems. Let us denote

$$\hat{B}_x(t) \quad [B_1x(t);B_2x(t);\ldots B_px(t)], \quad \hat{A}(t) \quad A+\sum_{i=1}^{p} u_{0i}(t)B_i \qquad (5)$$

So the linearized system for (1) is

$$\frac{d}{dt}x(t)=\hat{A}(t)x(t)+Cx(t-\tau)+\hat{B}_{x_0}(t)u(t) \qquad (6)$$

Then utilizing a result of Weiss [18], we have [12]:

Theorem 2: Let $x_0(t)=x(t;0,\varphi,u_0)$ be a trajectory of (1). Define the matrices $Q_i^j(t)$ via the equations

$$Q_0^0(t)=\hat{B}_{x_0}(t)$$

$$Q_i^j(t)=\dot{Q}_i^{j-1}(t)-\hat{A}(t+i\tau)Q_i^{j-1}(t)-CQ_{i-1}^{j-1}$$

$$i=1,\ldots m,\ j=i,\ldots,k-1$$

and $Q_i^j=0$ for $i<0$ or $i>j$

Let $Q(t)=[Q_0^0(t),\ldots Q_1^{k-1}(t),Q_1^1(t-\tau),\ldots Q_1^{k-1}(t-\tau),\ldots Q_m^{k-1}(t-m\cdot\tau)]$

Suppose there exist integer $k>0$ and time $t_1\epsilon[m\tau,(m+1)\tau)$ such that all the derivatives needed in the formation of Q exist and are continuous and rank $Q(t_1)=n$. Then the bilinear delay-differential system (1) has the local euclidean accessibility property along x_0 at t_1.

Finally we generalize the results of Sussman [15] to bilinear delay differential systems. Following [15] we let $\mathcal{U}(T)$=set of all measurable function defined on $[0,T]$ with values in the cube $\{(u_1,u_2,\ldots u_p): -1\le u_j\le 1, j=1,2,\ldots p\}$; $\mathcal{U}B(T)=\{u\epsilon\mathcal{U}(T): |u_i(t)|=1, i=1,\ldots p\}$; $\mathcal{U}BP(T)=\{u\epsilon\mathcal{U}B(T): u(t)$ is piecewise constant$\}$. According to whether we use controls from $\mathcal{U}(T)$ or $\mathcal{U}B(T)$ or $\mathcal{U}BP(T)$ we have for a given initial condition φ, the reachable sets $R(T,\varphi)$, $\mathbb{R}(T,\varphi)$, $RB(T,\varphi)$, $\mathbb{R}B(T,\varphi)$, $RBP(T,\varphi)$, $\mathbb{R}BP(T,\varphi)$.

The following theorem is then the analogue of Corollaries 1-3 of [15] in our setting, and its proof appears in [12, 14].

Theorem 3: The sets $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ are compact. The sets $RBP(T,\varphi)$ and $\mathbb{R}BP(T,\varphi)$ are dense in $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ respectively.

Consider now the reachable sets in function space $R_C(T,\varphi)$, $R_CB(T,\varphi)$, $R_CBP(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$, $\mathbb{R}_CB(T,\varphi)$, $\mathbb{R}_CBP(T,\varphi)$. We have then the analogue of Theorem 3 in function space.

Theorem 4: The sets $R_C(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$ are compact. The sets $R_CBP(T,\varphi)$, $\mathbb{R}_CBP(T,\varphi)$ are dense in $R_C(T,\varphi)$, and $\mathbb{R}_C(T,\varphi)$ respectively.

The proof appears in [12].

Finally we have the following result which provides an instance of a truly "Bang-Bang" theorem.

Theorem 5: If all the brackets $[B_i,B_j]$, $[A,B_i]$ vanish for all i, j then $RB(T,\varphi)$ and $\mathbb{R}B(T,\varphi)$ are closed. Moreover the sets $RBP(T,\varphi)$ and $\mathbb{R}BP(T,\varphi)$ are also closed.

The proof appears in [14], and utilizes the theorems of Liapunov, Aumann [16] and Halkin [17].

References

1. Brockett, R.W., "System theory on group manifolds and coset spaces", SIAM Journal on Control 10, 1974, pp. 265-284.
2. Sussmann, H.J. and V. Jurdjevic, J. of Diff. Eq., 12, 1972, pp. 95-116.
3. Brockett, R.W., "On the algebraic structure of bilinear systems", in Theory and Application of Variable Structure Systems, Academic Press, New York, 1972, pp. 153-168.
4. Brockett, R.W., "Lie algebras and lie groups in control theory", in Geometric Methods in Systems Theory, Reidel Pub. Co., the Netherlands, 1973.
5. Fliess, M., C.R. Acad. Sc. Paris, t. 280, Serie A, 1975, pp. 965-967.
6. Sussmann, H., Proc. of CNR-CISM Symposium on Algebraic System Theory, Udine, Italy, 1975.
7. Brockett, R.W., "Volterra series and geometric control theory", Proc. of 6th IFAC Congress, Boston, Mass., 1975.
8. Hale, J.K., Functional differential equations, Springer Verlag, New York, 1972.
9. Banks, H.T., J. Diff. Eqs., 5, 1969, pp. 399-409.
10. Delfour, M.C., C. McCalla and S.K. Mitter, SIAM J. Control, 13, 1975, pp. 48-88.
11. Bensoussan, A., M.C. Delfour and S.K. Mitter, Technical Rept. ESL-P-604, Dept. of Electrical Engineering, MIT.
12. Baras, J.S. and L. Hampton, "Bilinear delay-differential systems", Proc. of 1975 Conf. on Info. Sc. and Sys., April 1975, pp. 26-32.
13. Weiss, L., SIAM J. on Control, Vol. 5, No. 4, 1967, pp. 575-587.
14. Hampton III, L., "On the controllability of bilinear systems with delay", M.S. Thesis, Elec. Engr. Dept., Univ. of Md., May 1975.
15. Sussmann, H., "The "bang-bang" problem for certain control systems in GL(n,R)", SIAM J. Control, Vol. 10, No. 3, 1972, pp. 470-476.
16. Hermes, H. and J. LaSalle, Functional analysis and time optimal control, Ac. Press, 1969.
17. Halkin, H., "On a generalization of a theorem of Liapunov", J. Math. Anal. Appl., 10, 1965, pp. 325-329.
18. Weiss, L., IEEE Trans. Auto. Control, AC-15, 1970, pp. 443-444.