22nd International Symposium on
Mathematical Theory of Networks and Systems
July 11-15, 2016. Minneapolis, MN, USA

Consensus and Synchronized Periodicity in Nonlinear Delayed Networks

Christoforos Somarakis [1], John S. Baras [2], Nader Motee [1]

***Abstract*— In this paper, we investigate stability and convergence properties of a class of nonlinear delayed consensus networks. Using tools and techniques from functional differential equations, sufficient stability conditions with respect to a common state as well as estimates on the convergence rate are derived. We characterize the limit (consensus) state for time-invariant sub-classes of these networks. More importantly, we specify under what conditions a delayed network exhibits periodic synchronized solutions. We provide sufficient conditions for existence, uniqueness and stability of this interesting phenomenon that we illustrate with a simulation example.**

I. INTRODUCTION

Distributed cooperative dynamics have been vividly stimulating the attention of the engineering and applied mathematics communities for over a decade now. The most popular dynamics of this kind is the well-studied models of linear consensus algorithms.

More precisely, a linear consensus network involves $N \geq 2$ agents, in which each agent $i \in V := \{1, \dots, N\}$ is tagged with a state of interest, denoted by x_i. The state of each agent evolves under the following averaging schemes,

$$x_i(n+1) = \sum_j a_{ij} x_j(n), \quad \dot{x}_i(t) = \sum_j a_{ij} \big(x_j(t) - x_i(t)\big). \quad (1)$$

Certain connectivity criteria on the graph induced by the weight coefficients a_{ij} guarantee convergence of x_i in the long run to a common *consensus* state [3], [9], [10], [17]. Networks of type (1) have been extensively used in the literature to explain cooperative computation in robotic, social, biological, natural (flocking) networks (see for example [16] and references therein). There are active lines of research that aim at modifying the linear model (1) in order to incorporate more realistic operational conditions of real-world consensus networks, namely, delays and nonlinear couplings.

A. Nonlinear Models

Recent works investigate nonlinear variations of consensus networks supplying the literature with a fruitful of impressive results. Nonlinear versions of (1) exist in the literature primarily as extensions of the linear scheme, because they preserve its vital qualitative features [1], [7], [9], [11], [12]. In his seminal work [9], Moreau studied the generic system:

$$i \in V : \begin{cases} x_i(n+1) = f_i(n, x_1(n), \dots, x_N(n)), & n > n_0, \\ x_i(n_0) = x_i^0, & n = n_0 \end{cases} \quad (2)$$

where $n \in \mathbb{Z}_+$. Moreau built an asymptotic stability argument with respect to the set of consensus states, based on set-valued Lyapunov functions. He showed that agreement among agents emerges as $n \to \infty$, on condition that

$$x_i(n+1) \in \mathrm{co}\big\{x_1(n), \dots, x_N(n)\big\}, \ n \geq n_0. \quad (3)$$

The latter condition sets every agent's new state to lie in the interior of the convex hull of their neighbors' current states. A different approach was adopted, by the same researcher, for the continuous time linear version of (1) (see [8]). In that paper he provided a semi-rigorous proof of the contraction of the difference $\max_i x_i(t) - \min_i x_i(t)$ through time.

The nonlinear nature of (2) is essentially *sine qua non*. Carathéodory's Theorem assures that (3) is equivalent to the discrete (1) if $a_{ij} \geq 0$, $\sum_j a_{ij} \equiv 1$ [13]. These are, in fact, the connectivity conditions that characterize cooperative algorithms of type (1).

A first alternative of (1) is obtained assuming nonlinear couplings on agents' which possess certain passivity properties [1], [11] and this assumption has been used to rigorously study models from chemistry and social sciences [5], [4]. A second version is introduced in [12] as follows: For $i \in V$:

$$\begin{cases} \dot{x}_i(t) = \sum_j a_{ij}(t)\big(g_{ij}(x_j(t)) - g_{ij}(x_i(t))\big), & t \geq t_0 \\ x_i(t) = x_i^0, & t = t_0. \end{cases} \quad (4)$$

The authors prove convergence of solutions to a common constant in the presence of connectivity failures. Their work relies on Lyapunov stability and Invariance Principles in a differential inclusion theory; framework.

B. Delays

In real world situations the agents' access to information is not instant. Robots have terminals that may take some time to process data, exogenous conditions impose delays in the transmission of information from one node to another. Hence the phenomenon of delays plays a very important role in the reliability and performance of the consensus networks and it cannot be ignored. Generally speaking there are two categories of delays. The *processing* delays characterize the existence of delay while agent i processes its own value whereas the *propagation* delays that characterize the delay as agent i receives the transmitted states from the network.

* This work was partially supported by US AFOSR MURI grant FA9550-09-1-0538, NSF grant CNS-135655, NIST grant 70NANB11H148, NSF-ECCS-1202517, AFOSR-YIP FA9550-13-1-0158, ONR N00014-13-1-0636, and NSF CAREER ECCS-1454022

[1]C. Somarakis & N. Motee is with the Mechanical Eng. Dept., Lehigh University, Pennsylvania, USA csomarak@lehigh.edu, nam211@lehigh.edu

[2]J. S. Baras is with the Electrical & Computer Engineering Dept., The Institute For Systems Research, University of Maryland, College Park, USA baras@umd.edu

It is shown that while the former type can destabilize the dynamics, the latter type can only affect the performance of the network.

C. Contribution

The contribution of this paper is three-fold. We introduce delayed versions of (4) with multiple time-dependent propagation delays and study the existence and stability of consensus solutions through a fixed point theory argument. Our approach exploits a novel representation of the solution with respect to a nominal exponential stable delayed version of (1). The result associates the performance of the non-linear network with the strength of the non-linearity and the magnitude of the delays. We explain how a special type of (4) (called monotonic) with delays exhibits asymptotic stability with arbitrary multiple delays. Next, we characterize the consensus point in the autonomous case of networks with constant delays and we conclude by showing how non-linearity and delays can create periodic non-constant synchronized exponentially stable solutions. A simple example illustrates the latter result concluding the work.

II. Notations & Definitions

By $\mathbb{R}^N$ we denote the $N-$dimensional Euclidean space endowed with the norm $|\mathbf{y}| = \max_i |y_i|$. By $\mathbf{1}$ we understand the N-dimensional column vector of all ones. By $C^l([a,b],\mathbb{R}^N)$ we understand the Banach space of continuous functions mapping $[a,b]$ into $\mathbb{R}^N$ with the topology of uniform convergence that have $l \geq 0$ continuous derivatives. We set $C := C^0([-\tau,0],\mathbb{R}^N)$ for some $\tau > 0$ and we designate the norm $|\boldsymbol{\phi}| = \sup_{s\in[-\tau,0]} |\boldsymbol{\phi}(s)|$ for $\boldsymbol{\phi} \in C$. If $\mathbf{x} \in C([-\sigma-\tau,\sigma+\alpha],\mathbb{R}^N)$ then for $t \in [\sigma,\sigma+\alpha]$, we understand $\mathbf{x}_t \in C$ as $\mathbf{x}_t(s) = \mathbf{x}(t+s),\ s \in [-\tau,0]$. We recall L^1 the class of absolutely integrable functions. By $\frac{d}{dt}$ or "$\cdot$" we understand the right-hand side Dini derivative. A solution of an initial value problem through $t_0, \boldsymbol{\phi}$ is denoted by $\mathbf{x}(t_0,\boldsymbol{\phi})$ so that $\mathbf{x}_{t_0} = \boldsymbol{\phi}$. For any $\mathbf{x}_t \in C$ we define

$$W(\mathbf{x}_t) = [\min_{i\in V} \min_{s\in[-\tau,0]} x_i(t+s), \max_{i\in V} \max_{s\in[-\tau,0]} x_i(t+s)]$$
$$S(\mathbf{x}_t) = \max_{i\in V} \max_{s\in[-\tau,0]} x_i(t+s) - \min_{i\in V} \min_{s\in[-\tau,0]} x_i(t+s)$$

It can be easily shown that $S(\mathbf{x}_t)$ is a pseudo-metric: While $S(\mathbf{x}_t)$ is non-negative and it satisfies the triangular inequality, $S(\mathbf{x}_t) = 0$ implies

$$\{\mathbf{x}_t \in C : x_i(t+s) = x_j(t+s'),\ s,s' \in [-\tau,0],\ i,j \in V\}$$

that we denote by Δ. The symbol t_0 is reserved for the initial time. For an appropriate operator $R : [t_0 \times \infty) \times C \to \mathbb{R}^N$ the initial value problem

$$\begin{cases} \dot{\mathbf{y}}(t) = R(t)\mathbf{y}_t, & t \geq t_0 \\ \mathbf{y}(t) = \boldsymbol{\phi}(t), & t \in [t_0-\tau, t_0] \end{cases}$$

has a unique solution $\mathbf{y}(t,t_0,\boldsymbol{\phi}),\ t \in [t_0-\tau,\infty)$. Another representation is for fixed $t \geq t_0$, $\mathbf{y}_t(t_0,\boldsymbol{\phi})$ to be a member of C. To the initial value problem we associate a family of continuous linear operators $T(t,t_0) : C \to C$,

$$T(t,t_0)\boldsymbol{\phi} \to \mathbf{y}_t(t_0,\boldsymbol{\phi}).$$

The framework can incorporate general (possibly time-varying) delays, expressed as smooth functions of time. A delay between two agents i and j is defined as $\tau_{ij}(t) \in C^1\big([t_0,\infty),[0,\tau]\big)$. Also, for the sake of brevity, we adopt $\lambda_{ij}(t) = t - \tau_{ij}(t)$.

The framework for ordinary differential equations is recovered as a special case where $\tau = 0$.

A. Review of Stability Results for Linear Systems

The communication network can be rigorously described by a (possibly time varying) weighted graph $G_t = (V, E_t, W_t)$ where E_t denotes the set of edges and W_t the connectivity weight between nodes at every instant $t \geq t_0$. At this point we will review a stability result concerning the continuous time delayed version of (1), in the following initial value problem: For $i \in V$:,

$$\begin{cases} \dot{x}_i(t) = \sum_{j\in V} a_{ij}(t)\big(x_j(\lambda_{ij}(t)) - x_i(t)\big), & t \geq t_0 \\ x_i(t) = \phi_i(t), & t \in [t_0-\tau, t_0] \end{cases} \tag{5}$$

For this we will need a number of assumptions that characterize the connectivity conditions and the delays. The matrix representation of G_t is with the adjacency matrix $A(t)$, each element of which $a_{ij}(t)$ corresponds to the connectivity between nodes.

Assumption 2.1: [3] For any $i,j \in V$, $a_{ij} : [t_0,\infty) \to [0,\infty)$, integrable with the property that $t \geq t_0 : a_{ij}(t) \neq 0$ implies $a_{ij}(s) \geq \delta > 0$ for an ε interval of time.

Assumption 2.2: [8] The graph G_t time-varying attains the property there exists $B > 0$ such that $\int_t^{t+B} A(s)\,ds$ corresponds to a routed-out branching network.

Assumption 2.3: $\forall i,j \in V$, $\tau_{ij}(t) \in C^1([t_0,\infty),[0,\tau])$ such that $1 - \dot{\tau}_{ij}(t) > 0$.

Theorem 2.4: [16] Under Assumptions 2.1, 2.2 and 2.3, the solution $\mathbf{x}$ of (5) satisfies

$$S(\mathbf{x}_t) \leq \Gamma e^{-\gamma(t-t_0)} S(\boldsymbol{\phi}_{t_0})$$

for some $\Gamma, \gamma > 0$ that depend on the network parameters as described in the assumptions. In particular, $\exists\ k \in W(\boldsymbol{\phi}_{t_0})$ such that $\lim_{t\to\infty} x_i(t) \equiv k$ exponentially fast.

Remark 2.5: For $\tau < \infty$ set as the upper bound of the magnitude of the delays, $S(\mathbf{x}_t) \equiv 0$ implies that the system is at a constant consensus state.

Remark 2.6: Explicit estimates of Γ, γ in terms of the network parameters lie beyond the scope of this report. The interested reader is referred to [16].

III. Nonlinear Delayed Networks

We consider the scenario of a network of agents running a non-linear alrogithm of type (4) where each agent receives a delayed version of its neighboring agents' state. The corresponding initial value problem reads:

$$\begin{cases}\dot{x}_i(t) = \sum_j g_{ij}\big(t, x_j(\lambda_{ij}(t))\big) - \sum_j g_{ij}\big(t, x_i(t)\big),\ t \geq t_0 \\ x_i(t) = \phi_i(t),\ t \in [t_0 - \tau, t_0]\end{cases} \tag{6}$$

To the best of our knowledge, the literature lacks an effective framework to rigorously study the long term behavior of systems like (6) apart from special cases studied with very hard assumptions on $g_{ij}(\cdot)$ [12]. In this section, we will tackle this problem with the development of a fixed point theory argument by comparing (6) with (5) in which we will prove simultaneously, existence in the long run, uniqueness and stability of solutions, with explicit rate estimates.

Assumption 3.1: Assume that for all $i \neq j$ there are functions a_{ij} as in Assumption 2.1 that form a graph as in Assumption 2.2 and $k_{ij} : [t_0, \infty) \to [0, \infty)$ such that for any $y_1, y_2 \in \mathbb{R}$ and $t \geq t_0$,

$$\left| a_{ij}(t) - \int_0^1 g'_{ij}(t, qy_1 + (1-q)y_2))\, dq \right| \leq k_{ij}(t)$$

where $g'_{ij}(t,x) = \frac{\partial g_{ij}(t,x)}{\partial x}$ is assumed to exist and be integrable.

The above assumption generally establishes a growth estimate of g_{ij} with respect to a_{ij} and it is a hard Lipschitz condition. We are ready now to state the first result of this work.

Theorem 3.2: Let Assumptions 2.1, 2.2, 3.1 and 2.3, hold. Consider the initial value problem (6) with its solution $\mathbf{x}$. If

$$\sup_{t \geq t_0} D \int_{t_0}^{t} e^{-\gamma(t-\sigma)} k(\sigma)\, d\sigma < 1$$

where $D = 2(N-1)\Gamma e^{\gamma\tau}$, then there exists $\epsilon \in (0, \gamma)$, such that $\mathbf{x}_t \to \Delta$ as $t \to \infty$ exponentially fast with rate $\epsilon > 0$.

Proof: We observe that we can write (6) as

$$\begin{aligned}\dot{x}_i(t) = &\sum_j a_{ij}(t)\big(x_j(\lambda_{ij}(t)) - x_i(t)\big) \\ &+ \sum_j \big(\tilde{g}_{ij}(t, x_j(\lambda_{ij}(t)) - \tilde{g}_{ij}(t, x_i(t))\big)\end{aligned}$$

where $\tilde{g}_{ij}(t,y) = g_{ij}(t,y) - a_{ij}(t)y$. In vector form the systems of equations read

$$\dot{\mathbf{x}}(t) = -\mathbf{L}(t, \mathbf{x}_t) + \tilde{\mathbf{G}}(t, \mathbf{x}_t) \tag{7}$$

where $\mathbf{L}$ and $\mathbf{G}$ are $\mathbb{R}^N$-valued operators, acting on $[t_0, \infty) \times C$, with the i^{th} component to be $\sum_j [a_{ij}(t)x_i(t) - a_{ij}(t)x_j(\lambda_{1j}(t))]$ and $\sum_j [\tilde{g}_{ij}(t, x_j(\lambda_{1j}(t)) - \tilde{g}_{ij}(t, x_i(t))]$ respectively. Now, the linear network

$$\dot{\mathbf{z}}(t) = -\mathbf{L}(t, \mathbf{z}_t)$$

admits under Assumptions 2.1, 2.2, solutions that regardless $\boldsymbol{\phi}_{t_0}$, are exponentially stable with respect to Δ. Following [14] the solution $\mathbf{x}_t$ in terms of $\mathbf{z}_t = T(t, t_0)\boldsymbol{\phi}_{t_0}$ is

$$\mathbf{x}_t(t_0, \boldsymbol{\phi}_{t_0}) = T(t, t_0)\boldsymbol{\phi}_{t_0} + \int_{t_0}^{t} T(t, \sigma) Y \tilde{\mathbf{G}}(\sigma, \mathbf{x}_\sigma)\, d\sigma \tag{8}$$

where the last integral is with the understanding that

$$\mathbf{x}_t(t_0, \boldsymbol{\phi}_{t_0})(s) = [T(t, t_0)\boldsymbol{\phi}_{t_0}](s) + \int_{t_0}^{t} [T(t, \sigma)Y](s) \tilde{\mathbf{G}}(\sigma, \mathbf{x}_\sigma)\, d\sigma$$

for $s \in [-\tau, 0]$ and Y the $N \times N$ matrix valued function $Y(s) = 0$ for $s \in [-\tau, 0]$ and $Y = I$ for $s = 0$. The triangular inequality and Theorem 2.4 yield

$$\begin{aligned}S(\mathbf{x}_t) &\leq S(\mathbf{z}_t) + S\bigg(\int_{t_0}^{t} T(t, \sigma) Y \tilde{\mathbf{G}}(\sigma, \mathbf{x}_\sigma)\, d\sigma\bigg) \\ &\leq \Gamma e^{-\gamma(t-t_0)} S(\boldsymbol{\phi}_{t_0}) + S\bigg(\int_{t_0}^{t} T(t, \sigma) Y \tilde{\mathbf{G}}(\sigma, \mathbf{x}_\sigma)\, d\sigma\bigg)\end{aligned}$$

and the last term is bounded by $2\Gamma \int_{t_0}^{t} e^{-\gamma(t+s-\sigma)}(N-1)k(\sigma)S(\mathbf{x}_\sigma)\, d\sigma$, for $k(s) = \max_{ij} k_{ij}(s)$. So

$$S(\mathbf{x}_t) \leq \Gamma e^{-\gamma(t-t_0)} S(\boldsymbol{\phi}_{t_0}) + D \int_{t_0}^{t} e^{-\gamma(t-\sigma)} k(\sigma) S(\mathbf{x}_\sigma)\, d\sigma.$$

This inequality implies that $S(\mathbf{x}_t)$ is in fact upper bounded by the solution $q(t)$ of

$$q(t) = \Gamma e^{-\gamma(t-t_0)} q(t_0) + D \int_{t_0}^{t} e^{-\gamma(t-\sigma)} k(\sigma) q(\sigma)\, d\sigma \tag{9}$$

for $t \geq t_0$ and $q(t) = S(\boldsymbol{\phi}_{t_0})$.

1) Existence & uniqueness of a fixed point: Consider the following space of functions

$$\mathbb{M} = \big\{ y \in C^0\big([t_0, \infty), \mathbb{R}_+\big) : y(t_0) = S(\boldsymbol{\phi}_{t_0}), \sup_{t \geq t_0} e^{\epsilon t} |y(t)| < \infty \big\}$$

where ϵ as in the statement of the Theorem, together with the weighted metric

$$\rho(y_1, y_2) = \sup_{t \geq t_0} e^{\epsilon t} |y_1(t) - y_2(t)|$$

constitute a weighted complete metric space [2]. In this metric space $(\mathbb{M}, \rho)$ we will apply the Contraction Mapping Principle as follows: Define the operator $\mathcal{Q}y$ with $(\mathcal{Q}y)(t) = S(\boldsymbol{\phi}_{t_0})$ for $t = t_0$ and

$$(\mathcal{Q}y)(t) = \Gamma e^{-\gamma(t-t_0)} S(\boldsymbol{\phi}_{t_0}) + D \int_{t_0}^{t} e^{-\gamma(t-s)} k(s) y(s)\, ds$$

for $t \geq t_0$, and note for any $y \in \mathbb{M}$, $e^{\epsilon t}(\mathcal{Q}y)(t) \to 0$ if $\epsilon < \gamma$, as the first term clearly vanishes and the second term vanishes as the convolution of an L^1 function with a function that goes to zero. It is, finally, easy to see that $\mathcal{Q}$ is a contraction in $(\mathbb{M}, \rho)$ since

$$\rho(\mathcal{Q}y_1, \mathcal{Q}y_2) \leq D \sup_{t \geq t_0} e^{\epsilon t} \int_{t_0}^{t} e^{\gamma(t-s)} k(s) e^{-\epsilon s}\, ds \rho(y_1, y_2)$$

Then the imposed condition and Banach's Principle imply that there exists ϵ small enough so that $\mathcal{Q}$ is a contraction mapping in $(\mathbb{M}, \rho)$ hence it attains a unique solution in $\mathbb{M}$ ensuring that the unique solution (the one that majors $S(\mathbf{x}_t)$) vanishes exponentially fast with rate $\epsilon > 0$. ■

A. Monotonic couplings

Theorem 3.2 is admittedly a rather conservative stability result of (6) based on its relation to (5), the growth of the perturbation (Assumption 3.1) and the maximum delay. In particular, there is a strict interplay between D and k_{ij} for the stability result to apply. This approach is too restrictive because, heuristically speaking, we allowed "unregulated freedom" on the way g_{ij} is allowed to vary. A simple type of g_{ij}, makes the dynamics of (6) to mimic those of the linear case (5). In particular, we will ask the following *monotonic* condition[1].

Assumption 3.3: For all $i, j \in V$ it holds that $g_{ij} \in C^1\big([t_0,\infty) \times \mathbb{R}, \mathbb{R}\big)$ such that $g(t,x) \neq 0$ implies $\frac{\partial}{\partial x} g_{ij}(t,x) > 0$ uniformly in x.
This condition yields the following interesting corollary that we state here without proof.

Corollary 3.4: Let Assumptions of Theorem 3.2 hold together with Assumption 3.3. Then the solution $\mathbf{x}$ of (6) is exponentially stable with respect to Δ for arbitrary τ, with rate $\gamma > 0$.

B. Characterization of the consensus point

It is well known that when a consensus system attains time-invariant parameters there is a closed form solution of the consensus point. In the absence of delays and constant connectivity weights the solutions converge to $\mathbf{w}^T\mathbf{x}^0$ where $\mathbf{w} \in \mathbb{R}^N$ satisfies $\mathbf{w}^T L = 0$, $w_i \geq 0$, $\sum_i w_i = 1$, for L the laplacian matrix of the network. Time-dependent parameters (connection weights or delays) make generally the expression of the consensus point in an explicit form inevitable. Non-autonomous dynamics affect constantly the system's solution in a way that cannot simply depend on the initial conditions. In this section we will consider and autonomous version of (6) under a supplementary symmetry assumption:

Assumption 3.5: For the nonlinear functions g_{ij} it holds that $g_{ij} \in C^1(\mathbb{R},\mathbb{R})$ with the property $g'_{ij} = g'_{ji}$.

For the main result we will need the next technical lemma:

Lemma 3.6: Let Assumption 3.3 hold. Given $\phi_i \in C^0([-\tau,0],\mathbb{R})$, $i = 1,\dots,N$ then $\exists\,!\ c \in W(\boldsymbol{\phi})$ to satisfy

$$c = \sum_i \alpha_i \phi_i(0) + \sum_{i,j} \beta_i \bigg(\int_{-\tau_{ij}}^0 g_{ij}(\phi_j(s))\,ds - \tau_{ij} g_{ij}(c) \bigg)$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^N \geq 0$ with $\sum_i \alpha_i = \sum_i \beta_i = 1$.

Proof: Define the function $J : W(\boldsymbol{\phi}) \to \mathbb{R}$

$$J(c) = c - \sum_i \alpha_i \phi_i(0) - \sum_{i,j} \beta_i \bigg(\int_{-\tau}^0 g_{ij}(\phi_j(s))\,ds + g_{ij}(c) \bigg)$$

We begin by excluding the trivial cases. This is for $W(\boldsymbol{\phi})$ being a singleton, i.e. $W(\boldsymbol{\phi}) = \{c\}$ and hence $\phi_i \equiv c$ and automatically $J \equiv 0$. If $W(\boldsymbol{\phi})$ is not a singleton then we take

$$c_1 := \min_{i\in V} \min_{s\in[-\tau,0]} \phi_i(s) < \max_{i\in V} \max_{s\in[-\tau,0]} \phi_i(s) =: c_2$$

by continuity of ϕ_i and g_{ij} we conclude that

$$c_1 \leq \phi_i(0) \quad \text{and} \quad g_{ij}(c) \leq g_{ij}(\phi_i(s))$$

but with some i, j such that $g_{ij}(c) < g_{ij}(\phi(s))$ for some $s \in [-\tau, 0]$. Consequently, $J(c_1) < 0$ and similar analysis will yield $J(c_2) > 0$. By the theorem of Bolzano there exists $c \in W$ such that $J(c) = 0$. The uniqueness of c follows from the fact that $J' = 1 + \sum_{i,j} \beta_i g'_{ij}(c) > 0$. ∎

Theorem 3.7: Consider the system (6) with $g_{ij}(t,x) = g_{ij}(x)$ and $\tau_{ij}(t) \equiv \tau_{ij}$. Let Assumptions 3.3 and 3.5 hold and the nonlinear network is simply connected. The solution $\mathbf{x} = \mathbf{x}(t,t_0,\boldsymbol{\phi})$, $t \geq t_0$ of (6) converges to the unique solution of

$$c = \sum_j \alpha_j \phi_j(0) + \sum_{i,j} \frac{1}{N} \int_{-\tau_{ij}}^0 g_{ij}\big(\phi_j(s)\big)\,ds - \sum_{i,j} \frac{1}{N} \tau_{ij} g_{ij}(c)$$

exponentially fast with rates dictated by Theorem 2.4.

Proof: [Sketch] It suffices to show that $\mathbf{x}$ indeed converges to a point satisfying the aforementioned nonlinear algebraic equation. We rewrite (6) as follows:

$$\dot{x}_i(t) = \sum_j \Bigg(g_{ij}(x_j) - g_{ij}(x_i) - \frac{d}{dt} \int_{t-\tau_{ij}}^t \big[g_{ij}\big(x_j(s)\big) - g_{ij}(c)\big]\,ds \Bigg)$$

Consider the solution $\mathbf{y}$ of

$$i \in V : \begin{cases} \dot{y}_i = \sum_j g_{ij}(y_j) - \sum_j g_{ij}(y_i), & t > 0 \\ y_i(t) = \phi_i^0, & t = 0 \end{cases} \tag{10}$$

In vector form (10) reads,

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}), \qquad \mathbf{y}(0) = \boldsymbol{\phi}_0$$

For $t \geq s \geq 0$, $\mathbf{V}(s) := \mathbf{y}\big(t,s,\mathbf{x}(s)\big)$ and differentiate with respect to s

$$\frac{d}{ds}\mathbf{V}(s) = -\frac{\partial \mathbf{y}\big(t,s,\mathbf{x}(s)\big)}{\partial \boldsymbol{\xi}} \frac{d}{ds}\mathbf{H}(\mathbf{x}_s)$$

where $\mathbf{H}(\mathbf{x}_s)$ is a vector with the i^{th} element $\sum_j \int_{s-\tau_{ij}}^s \big[g_{ij}(x_j(\sigma)) - g_{ij}(c)\big]\,d\sigma$. Note that $\frac{\partial \mathbf{y}(t,s,\mathbf{x}(s))}{\partial \boldsymbol{\xi}}$ is the principal matrix solution of the following linear non-autonomous system

$$\dot{\mathbf{z}} = \mathbf{G}'\big(\mathbf{y}(t,s,\mathbf{x}(s))\big)\mathbf{z}.$$

This is a consensus network with symmetric weights for which it is very well known that regardless of $\mathbf{y}(t,s,\mathbf{x}(s))$, $\mathbf{z}(t) \to \frac{\mathbf{1}\mathbf{1}^T}{N}\mathbf{z}^0$ exponentially fast. Next, we integrate from 0 to t to obtain the following expression for the solution of (6)

$$\mathbf{x}(t,0,\boldsymbol{\phi}) = \mathbf{y}(t,0,\boldsymbol{\phi}^0) - \int_0^t \frac{\partial \mathbf{y}\big(t,s,\mathbf{x}(s)\big)}{\partial \boldsymbol{\xi}} \frac{d}{ds}\mathbf{H}(\mathbf{x}_s)\,ds$$

Integration by parts and change of the order of integration in the last equation yields as $t \to \infty$

[1]The term "monotone" is due to H. Smith [15]

$$\mathbf{1}c = \mathbf{1}\sum_j \alpha_j\phi_j(0) + \mathbf{1}\sum_{i,j}\frac{1}{N}\left(\int_{-\tau_{ij}}^0 g_{ij}\big(\phi_j(s)\big)\,ds - \tau_{ij}g_{ij}(c)\right)$$

which, by Lemma 3.6, we know that it attains a unique solution in $W(\boldsymbol{\phi}_0)$. ■

IV. Periodic synchronized solutions

Studies in consensus systems with propagation delayed information confirm that the effect of delays appear only on the performance of the network. In this section we will show that the effect of propagation delays in cooperative systems may cause more complex behavior than simply weakening the rate of convergence to consensus. Indeed, whenever the dynamics are nonlinear and the type of delays is distributed, there seems to be a possibility of a non-trivial periodic solution. The period of the solution is directly connected to the magnitude of the delay. We focus our discussion on both periodic and synchronized solutions.

Definition 4.1: A function $\mathbf{y} \in C([t_0,\infty),\mathbb{R}^N)$ is *synchronized* if $|\mathbf{y}| < \infty$ and $S(\mathbf{y}(t)) \equiv 0$.

This is an extended concept of agreement that basically accepts consensus along a non-trivial orbit. The forms of networks that exhibit such behavior are substantially different from the previous ones. Consider the following initial value problem; For $i \in V$:

$$\begin{cases}\dot{x}_i(t) = \sum_j \left[\int_{-\tau}^0 g_{ij}\big(t,s,x_j(t+s)\big)p(t,s)\,ds - g_{ij}\big(t,t,x_i(t)\big)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \ t \ge t_0 \\ x_i(t) = \phi_i(t) \qquad\qquad\qquad\qquad , \ t \in [t_0-\tau, t_0]\end{cases} \tag{11}$$

where $p \in C^0([t_0,\infty)\times[-\tau,0],\mathbb{R}_+)$ has the property

$$\int_{-\tau}^0 p(t,s)\,ds = 1, \qquad t \ge 0. \tag{12}$$

The working hypothesis in the conventional consensus networks models agent i to receive the signal with the state x_j from agent j with a coupling weight which suffers from no processing delay. This condition would makes sense only if the particular rate is a parameter controlled exclusively by i. Otherwise, if the information on the coupling rate is also transmitted from j should suffer from delays. We will show here that if this is the case, periodic solutions can occur. We study the generic scenario where uncertainty is put in an interval of possible delays. This is typically expressed via a distribution function that smoothly weights the different possible delays.

A. Delay induced synchronization

Exhaustive simulations have suggested the following modification of the consensus network:

$$\begin{cases}\dot{x}_i(t) = -\sum_j g_{ij}\big(t,x_i(t)\big) + \sum_j \int_{t-T}^t g_{ij}\big(s,x_j(s)\big)p(s-t)\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ,t \ge 0 \\ x_i(t) = \phi_i(t), \ t \in [-T,0]\end{cases} \tag{13}$$

where p is a distributed delay satisfying (12) with $\tau = T$. The conditions we are imposing on g_{ij} are significantly harder than the ones considered so far.

Assumption 4.2: For all $i,j \in V,\ t \ge 0,\ x \in \mathbb{R}$, the following properties hold:

(i.) $g_{ij}(t,x) > 0$, uniformly in t, if and only if j is connected to i and zero otherwise.
(ii.) $g_{ij}(t+T,x) = g_{ij}(t,x)$,
(iii.) $\frac{\partial}{\partial x}g_{ij}(t,x) \in C^0\big([0,\infty)\times\mathbb{R},[\underline{K},\overline{K}]\big)$ for $(i,j) \in E$ and some $0 < \underline{K} \le \overline{K} < \infty$, independent of t
(iv.) $\sum_j g_{ij}(t,x)$ is independent of i.

Assumption 4.3: The connectivity graph is static and there exists $j \in V : g_{ij} \ne 0$ for all $i \in V\backslash\{j\}$.

Proposition 4.4: Let Assumption 4.2 hold. If

$$\overline{K}\int_{-T}^0 p(s)(-s)\,ds < 1,$$

then there exists a unique synchronized periodic solution of (13) with period T. The solution is constant only if there is c such that

$$\sum_j g_{ij}(t,c) = \int_{-T}^0 p(s)\sum_j g_{ij}(t+s,c)\,ds.$$

Proof: [Sketch] We begin with the second statement. If $\mathbf{x}$ is synchronized and $T-$periodic, i.e. $\mathbf{x}(t) = \big(x_1(t),\dots,x_N(t)\big) = \big(\zeta(t),\dots,\zeta(t)\big)$ for some appropriate function. By Assumption 4.2$(v.)$,

$$\dot{\zeta}(t) = -\sum_j g_{ij}\big(t,\zeta(t)\big) + \sum_j\int_{t-T}^t g_{ij}\big(s,\zeta(s)\big)p(s-t)\,ds$$

independent of i. If $\zeta(t) \equiv c$ for some $c \in \mathbb{R}$, then the last equation reads

$$\begin{aligned}0 &= -\sum_j g_{ij}(t,c) + \sum_j\int_{t-T}^t g_{ij}(s,c)p(s-t)\,ds \\ &= -\sum_j g_{ij}(t,c) + \int_{-T}^0 p(s)\sum_j g_{ij}(t+s,c)\,ds\end{aligned}$$

We proceed by proving the existence and uniqueness of a periodic solution $\mathbf{x}(t) = \mathbf{1}\zeta(t)$. Adding and subtracting $g_{ij}\big(t,x_j(t)\big)$ we arrive at the following equivalent equation for x_i

$$\begin{aligned}\dot{x}_i(t) =& \sum_j g_{ij}\big(t,x_j(t)\big) - \sum_j g_{ij}\big(t,x_i(t)\big) \\ &- \sum_j \frac{d}{dt}\int_{-T}^0 p(s)\int_{t+s}^t g_{ij}\big(w,x_j(w)\big)\,dwds\end{aligned}$$

We follow the steps of the proof of Theorem 3.7. We set $l(s) = \mathbf{x}\big(t,s,\mathbf{z}(s)\big)$, differentiate with respect to s and integrate from 0 to t and we express the solution $\mathbf{x}(t) = \mathbf{x}(t,0,\boldsymbol{\phi})$ of (13) as follows:

$$\begin{aligned}&\mathbf{x}(t,0,\boldsymbol{\phi}) = \mathbf{z}(t,0,\boldsymbol{\phi}) - \\ &-\int_0^t \frac{\partial\mathbf{z}(t,s,\mathbf{x}(s))}{\partial\boldsymbol{\xi}}\frac{d}{ds}\int_{-T}^0 p(q)\int_{s+q}^s \mathbf{G}(w,\mathbf{x}(w))\,dwdqds\end{aligned} \tag{14}$$

Let $(\mathbb{S}, |\cdot|)$ be the Banach space of continuous T-periodic synchronized functions. Define the operator $\mathcal{P} : \mathbb{S} \to \mathbb{B}$ as follows:

$$(\mathcal{P}\mathbf{x})(t) = \mathbf{x}_{(14)}(t)$$

where $\mathbf{x}_{(14)}(t)$ is the right hand-side of (14) as it was expressed above. Interestingly enough, $\mathcal{Q}$, restricted to $\mathbb{S}$, becomes particularly simplified. Indeed, for $\mathbf{x} \in \mathbb{S}$ under Assumption 4.2:

1) $\mathbf{z}(t, 0, \boldsymbol{\phi}) \equiv \boldsymbol{\phi}$,
2) $\mathbf{G}\big(w, \mathbf{x}(w)\big) \in \Delta$ for any fixed $w \geq -T$.

Consequently, the principal matrix $\frac{\partial \mathbf{z}}{\partial \boldsymbol{\xi}}$ acts on Δ and thus has no effect on any such element of Δ. So we are left with

$$\begin{aligned}(\mathcal{P}\mathbf{x})(t) = \boldsymbol{\phi}(0) &+ \int_{-T}^{0} p(q) \int_{q}^{0} \mathbf{G}\big(w, \boldsymbol{\phi}(w)\big)\, dwdq \\ &- \int_{-T}^{0} p(q) \int_{t+q}^{t} \mathbf{G}\big(w, \mathbf{x}(w)\big)\, dwdq\end{aligned}$$

and it is easy to see that $(\mathcal{P}\mathbf{x})(t+T) = (\mathcal{P}\mathbf{x})(t)$.

Finally, under the stated condition $\mathcal{Q}$ becomes a contraction under the metric $\rho(\mathbf{x}, \mathbf{y}) = \sup_t \max_i |x_i(t) - y_i(t)|$ and the Contraction Mapping Principle applies to prove existence and uniqueness of a fixed point in $\mathbb{S}$, concluding the proof. ■

Proposition 4.4 states a sufficient condition for existence and uniqueness of a periodic synchronized solution. We will see now that this condition actually suffices for the local asymptotic stability of $\mathbf{1}\zeta$.

Theorem 4.5: Let Assumptions 4.2 and 4.3 hold. The synchronized solution $\mathbf{1}\zeta(t)$ of Proposition 4.4 is locally exponentially stable if

$$\sup_{t \geq 0} \int_{-T}^{0} \int_{t+s}^{t} \left[\sum_j \frac{\partial g_{ij}(w, \zeta(w))}{\partial x} \right] dwp(s)\, ds < 1.$$

Proof: [Sketch]We will make a first variation orbital stability argument. Assumption 4.2 implies that the right hand-side of (13) has continuous first order partial derivatives globally. Let $\mathbf{1}\zeta(t)$ be the T-periodic solution of (13) defined from Proposition 4.4 and $\mathbf{x}(t, 0, \boldsymbol{\phi})$ a solution of (13) so that $\boldsymbol{\phi}$ is in the vicinity of $\mathbf{1}\zeta$. For $t \geq 0$, set $z(t) = \max_i |x_i(t) - \zeta(t)|$. If $x_i(t) - \zeta(t) \geq 0$ then Taylor's theorem yields

$$\begin{aligned}\frac{dt}{dt} z(t) \leq &- \sum_j \frac{\partial g_{ij}\big(t, \zeta(t)\big)}{\partial x} z(t) \\ &+ \int_{t-T}^{t} \sum_j \frac{\partial g_{ij}\big(s, \zeta(s)\big)}{\partial x} z(s) p(s-t)\, ds + o(|z|).\end{aligned}$$

A similar argumentation for $x_i(t) - \zeta(t) < 0$ yields the same upper bound for $\dot{z}(t)$. Consequently, for initial data near the periodic orbit we omit the higher order terms $o(|z|)$ and observe that $\dot{z}(t) \leq \dot{q}(t)$ for

$$\begin{aligned}\dot{q}(t) = &- \sum_j \frac{\partial g_{ij}\big(t, \zeta(t)\big)}{\partial x} q(t) \\ &+ \int_{t-T}^{t} \sum_j \frac{\partial g_{ij}\big(s, \zeta(s)\big)}{\partial x} q(s) p(s-t)\, ds\end{aligned}$$

In view of (12) we write

$$\frac{d}{dt} q(t) = -\frac{d}{dt} \int_{-T}^{0} \int_{t+s}^{t} \Big[\sum_j \frac{\partial g_{ij}\big(w, \zeta(w)\big)}{\partial x} \Big] q(w)\, dwp(s)\, ds$$

and this will yield

$$q(t) = q_0 - \int_{-T}^{0} \int_{t+s}^{t} \Big[\sum_j \frac{\partial g_{ij}\big(w, \zeta(w)\big)}{\partial x} \Big] q(w)\, dwp(s)\, ds \tag{15}$$

with $q_0 = q(0) + \int_{-T}^{0} \int_{s}^{0} \big[\sum_j \frac{\partial g_{ij}(w, \zeta(w))}{\partial x} \big] q(w)\, dwp(s)\, ds$. For $z(\cdot)$ as defined above and $\chi > 0$ we consider the functional space

$$\begin{aligned}\mathbb{V} = \big\{ v(t) \in C^0([-T, \infty), \mathbb{R}) :\ & v(s) = z(s)|_{s \in [-T, 0]}, \\ & \sup_{t \geq 0} e^{\chi t} |v(t)| < \infty \big\}\end{aligned}$$

which, together with the weighted metric, $\rho(v_1, v_2) = \sup_t e^{\chi t} |v_1(t) - v_2(t)|$ constitutes a complete metric space [2]. Define the mapping $\mathcal{Q} : \mathbb{V} \to \mathbb{B}$

$$(\mathcal{Q}v) = \begin{cases} z(t), & t \in [-T, 0] \\ q_{(15)}(t), & t \geq 0 \end{cases}$$

where $q_{(15)}(t)$ stands for the right hand-side of (15). It is easy to see that $\mathcal{Q} : \mathbb{V} \to \mathbb{V}$ and under the imposed condition one can pick $\chi > 0$ small enough so that

$$\sup_{t \geq 0} e^{\chi t} \int_{-T}^{0} \int_{t+s}^{t} \left[\sum_j \frac{\partial g_{ij}(w, \zeta(w))}{\partial x} \right] e^{-\chi w}\, dwp(s)\, ds < 1.$$

Then $\mathcal{Q}$ becomes a contraction in $\mathbb{V}$ and the Contraction Mapping Principle applies to ensure a unique fixed point. So $q(t)$ converges to 0 exponentially fast and so does $z(t)$. ■

V. A Simulation Example

Due to space limitations we will illustrate only Theorem 4.5 with a simple simulation run over a small network. Consider the 3×3 network

$$\dot{x}_i = -\sum_{j=1}^{3} a_{ij}(t) g_{ij}\big(x_i\big) + \int_{t-1}^{t} \sum_{j=1}^{3} a_{ij}(s) g_{ij}\big(x_j(s)\big) p(s-t)\, ds$$

with

$$G(x) = \bar{g} \begin{bmatrix} 0 & 0.01x + \frac{x^3}{1+x^2} & 3x + \sin^2(x) \\ 0.01x + \frac{x^3}{1+x^2} & 0 & 3x + \sin^2(x) \\ 3x + \sin^2(x) & 0.01x + \frac{x^3}{1+x^2} & 0 \end{bmatrix}$$

for some $\bar{g} > 0$ and

$$A(t) = \begin{bmatrix} 0 & 2 + \sin(2\pi t) & 3 + \sin(4\pi t) \\ 2 + \sin(2\pi t) & 0 & 3 + \sin(4\pi t) \\ 3 + \sin(4\pi t) & 2 + \sin(2\pi t) & 0 \end{bmatrix}.$$

It can be easily verified that the system satisfies Assumptions 4.2 and 4.3. Moreover, if $p(s) \equiv 1$, (12) is satisfied, as well. Choosing $\bar{g} < \frac{1}{6.14}$ both the conditions of Proposition

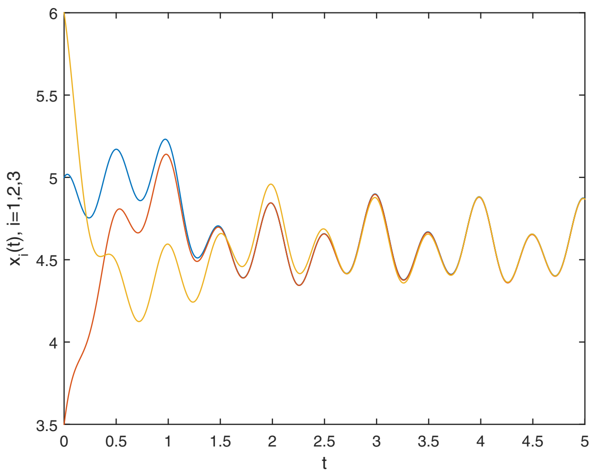

Fig. 1. Simulation run with the `ddesd` routine in `MATLAB`.

4.4 and Theorem 4.5 hold and this means that there is a unique solution that is exponentially stable with rate $\chi = 0.0015$. See Figure 1 for a numerical calculation of the solution. A simulation study of the solutions of the network reveals that the upper bound of $\bar{g}$ is conservative. Indeed the monotonicity of g_{ij} suggests that periodic solutions exist and are asymptotically stable for arbitrary values of $\bar{g}$ and arbitrary initial data.

VI. Concluding Remarks

In this paper we considered classes of nonlinear delayed extension of consensus networks developing nonlinear methods for the characterization of the long-term behavior of the its solutions. Sufficient conditions for asymptotic stability depend not only on the connectivity regime between nodes but also on the type of the coupling non-linearities as well as the magnitude of the delays in the network. Whenever the nonlinear systems incorporate monotonic features hard stability conditions are replaced by the standard ones from linear theory.

The autonomous case also provides information on the consensus point. The monotonicity condition implies that the consensus point serves as a unique solution of a nonlinear algebraic equation that depends on the connectivity of the network as well as the (constant) delays.

In the last part of the work section we pointed out that although the standard type in synchronization solutions of consensus systems is the constant solutions, in an interesting turn of events we showed that for the special type of distributed delays, nonlinearity provides an alternative: the existence and asymptotic stability of synchronized yet non-constant periodic solutions. We proved the existence and uniqueness of a periodic solution with a fixed point theorem approach and its local stability using the classic variational argument. Our objective was to merely point to this a direction of dynamic behavior thus the very strong and simplifying assumptions of Theorem 4.5 and open the discussion with the introduction of the standard framework and the first scratching of the surface of this topic.

Despite several advantages of non-linear variational techniques we conclude this work by discussing the drawback of too strong assumptions of Theorem 3.2. The key reason is that the metric $S(\mathbf{x}_t)$ is by its nature very conservative because it considers τ-interval of solutions, hence it provides estimates that grow exponentially fast with τ. In addition, the derivation of the solution operator in the proof of the theorem is generally crude. One would follow more involved methods discussed in [6] for the solution representation in terms of $S(\mathbf{x}_t)$ provided that one would prove that the spread operator $S : C \to \mathbb{R}^N$ attains an, at least integrable, Fréchet derivative.

It is clear that the discussed drawbacks of our approach constitutes the challenges that pave the way for future research work.

References

[1] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, 2007.
[2] T.A. Burton. *Stability by Fixed Point Theory for Functional Differential Equations*. Dover, 2006.
[3] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.
[4] U. Krause. A discrete nonlinear and non-autonomous model of consensus formation. *Communications in difference equations*, pages 227–236, 2000.
[5] Y. Kuramoto. *Chemical Oscillations, Waves, and Turbulence*. Springer-Verlag, 1984.
[6] V. Lakshminkantham and Deo S.G. *Method of Variation of Parameters for Dynamic Systems*, volume 1. Gordon and Breach Science, 1998.
[7] Jan Lorenz and Dirk A. Lorenz. On conditions for convergence to consensus. *IEEE Trans. Automat. Contr.*, 55(7):1651–1656, 2010.
[8] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *43rd IEEE Conference on Decision and Control*. IEEE, 2004.
[9] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(02):169–182, 2005.
[10] R. Olfati-Saber and R. M. Murray. Consensus problems in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(09), 2004.
[11] A. Papachristodoulou, A. Jadbabaie, and U. Mnz. Effects of delay in multi-agent consensus and oscillator synchronization. *IEEE Trans. Automat. Contr.*, 55(6):1471–1477, 2010.
[12] H. Qing and W. M. Haddad. Distributed nonlinear control algorithms for network consensus. *Automatica*, 44(9):2375 – 2381, 2008.
[13] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1997.
[14] G. A. Shanholt. A nonlinear variation-of-constants formula for functional differential equations. *Math. Systems Theory*, 6:343–352, 1973.
[15] H.L Smith. *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, volume 41. AMS, 1995.
[16] C. Somarakis and J. S. Baras. A general framework for consensus systems. Technical Report doi:10.13016/M2WK77, Digital Repository at the University of Maryland, 2015. http://hdl.handle.net/1903/16324.
[17] J. Tsitsiklis, D. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986.