SIAM J. CONTROL OPTIM.
Vol. 58, No. 3, pp. 1626–1651

STABLE CONSENSUS DECISION MAKING FOR SPATIALLY DISTRIBUTED MULTIAGENT SYSTEMS WITH MULTIPLE LEADERS*

ZHIXIN LIU†, LIN WANG‡, DAOYI DONG§, AND JOHN S. BARAS¶

Abstract. This paper considers the consensus decision-making problem of spatially distributed multiagent systems with multiple leaders, where the leaders have the preference about the destination, while the followers have no such preference. The agents have limited capability to sense the movement information of their local neighbors determined via the spatial relative distance, and make movement decisions accordingly. The objective is to guide the followers to move with the same preferred direction as leaders have (i.e., reaching a stable consensus). We provide detailed analysis for the information transfer between the subgroups of leaders and followers, and establish quantitative results on the proportion of leaders needed to reach the stable consensus for two cases: leaders with the same preference and leaders with different preferences. For the system where the leaders have the same preference, we provide an upper bound (impossibility theorem) and a lower bound (sufficient condition) for the proportion of leaders needed in reaching the stable consensus. When the leaders have two different preferences, we provide a sufficient condition and a necessary condition on the proportions of two subgroups of leaders to achieve stable consensus decisions.

Key words. multiagent systems, leader-follower model, proportion, stable consensus decision

AMS subject classifications. 92D50, 91C20, 93C55

DOI. 10.1137/16M1063290

1. Introduction.

The study of multiagent systems (MASs), such as bird flocks, fish schools, people in panic, cooperation among species, opinion dynamics, has attracted much attention from researchers in diverse fields; see [1], [2], [3], [4], [5], [6], [7] among many others. Consensus, meaning that all agents reach the same state, is a basic collective behavior. It has been extensively investigated over the past decade due to its close connections with some natural phenomena (e.g., flocking in biological systems and magnetization in physical systems), and its wide applications in engineering systems (cf. [8], [9], [10], [11], [12], [13]).

Some theoretical results for the consensus analysis of MASs have been given in, e.g., [14], [15], [16]. In most existing work, we see that reaching consensus needs the system parameters to satisfy certain conditions on neighbor graphs or model parameters, and the states resulting from the self-organization may not be what we expect

*Received by the editors March 16, 2016; accepted for publication (in revised form) April 1, 2020; published electronically June 18, 2020.
https://doi.org/10.1137/16M1063290
Funding: This work was supported by the National Key R&D Program of China under grant 2018YFA0703800, by the National Natural Science Foundation of China under grants 11688101, 61621003, and 61873167, and by the Australian Research Council's Discovery Projects funding scheme under project DP130101658.
†Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, CAS, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, People's Republic of China (lzx@amss.ac.cn).
‡Department of Automation, Shanghai Jiaotong University, and the Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai 200240, People's Republic of China China (wanglin@ sjtu.edu.cn).
§School of Engineering and Information Technology, University of New South Wales, Canberra, 2600 Australia (daoyidong@gmail.com).
¶Electrical Engineering Department and the Institute for Systems Research, University of Maryland, College Park, MD 20742 (baras@umd.edu).

even if the system can reach consensus. However, in some practical situations, we require that the agents reach the desired consensus states, for example, leading the people in panic to the safe place, guiding the bird flocks to avoid collisions with obstacles. How to induce an MAS to the desired behavior is one of the important topics in the investigation of MASs. In many biological systems and social networks, there are a small number of special agents (leaders) with the preference about the moving direction and they can help guide the whole population (cf. [17], [18]), which inspires us to intervene in an MAS by leaders. This paper will quantitatively investigate the proportion of leaders needed for the consensus decision making. Some theoretical results for the MASs with one leader have been presented in the fields of control and robotics. For example, Jadbabaie, Lin, and Morse [14] considered the MAS where the leader has a constant signal. The proportional and derivative-like discrete-time consensus algorithm was proposed in [20] to track the leader with a time-varying signal. The leader-follower model where the input of the leader is unknown was considered in [21]. The consensus of the leader-follower model with second-order dynamics and the Lagrange dynamics has also been considered (cf. [8], [22], [23]). Most of theses results require the neighbor graphs to satisfy a certain connectivity assumption. We remark that the connectivity of static graphs has been investigated in wireless sensor networks; see [24] and [25]. But these results are not suitable to deal with the system considered in this paper since here the neighbor graphs are determined via the states of the system, and thus coupled with the states of the system and dynamically change over time. How to verify or guarantee the connectivity conditions of the dynamical system has been a challenging issue. For continuous-time dynamics, the potential function method is often used to design the control law of agents such that they can track the reference signal of leaders; see [29], [30], [31] with different design of potential functions. While for the discrete-time systems with first-order and second-order dynamics, the connectivity maintenance problem has been investigated in, e.g., [26], [27], [28], where the distributed algorithms are designed such that the positions of the agents converge to a common point. These results are not feasible to deal with the consensus decision-making problem under consideration since the local rules of leaders and followers cannot be designed, and the number of leaders needed for consensus is not concerned.

A further development is to introduce multiple leaders into the system. MASs with multiple leaders widely exist in biological systems and social networks (cf. [17], [18], [33]). The investigation of such systems is generally more complicated and challenging. Couzin et al. [19] proposed a spatially distributed leader-follower model to study the influence of the proportion of leaders to consensus decision making, and they presented some qualitative results by computer simulations. When the leaders have the same preference, the larger the population size, the smaller the proportion of leaders required for the stable consensus. If there are two subgroups of leaders where each has one preferred direction, and the number of one subgroup of leaders is larger than that of the other, then the followers track the preferred directions of leaders who are in a majority, even if the majority is small. Motivated by this work, some theoretical results for MASs with multiple leaders have been established. Gustavi et al. [32] presented a sufficient condition on the number of leaders to guarantee the agreement of the positions for fully connected neighbor graphs. Nabet et al. [34] considered the MASs with two subgroups of leaders, each having one preferred direction. Some theoretical results concerning the consensus decision making have been presented under the assumptions that there is no autonomous agent in the group, and all agents are coupled together by all-to-all communication. Leonard et al. [35]

relaxed these limitations, and provided the threshold of disagreement of preferences for the stable consensus. Though these theoretical results were obtained, quantitative analysis for the influence of the proportion of leaders on the consensus decision making is still lack. Liu, Han, and Hu [36] presented some preliminary results and established a lower bound on the proportion of leaders for the expected consensus. Here, we further present a necessary condition that can characterize the impossibility of the stable consensus decision making. In addition, in many social networks and biological systems, there may exist several classes of leaders in an MAS. For example, in a large scale social event, policemen could be one class of leaders who have the preference to guide the people (followers) to a safe place, while terrorists could be another class of leaders who have the preference to guide the people to a dangerous place. In an opinion network, there may exist several classes of leaders from different parties who expect to guide the voters to support them. A significant issue is to quantitatively investigate how the proportions of leaders affect the consensus decisions. For these cases, conflicts of preferences about the destination confuse the followers in consensus decision making. To the best of our knowledge, there are no quantitatively theoretical results on how the proportions of leaders affect the dynamical behavior of followers in the literature. In comparison with the previous work [36], this paper establishes an impossibility theorem for the proportion of leaders needed for the case where the leaders have the same preference, and presents new quantitative results on the proportions of leaders for the stable consensus when the leaders have different preferences. These results may have potential applications in some real systems. For example, the people will be in panic when they are in jamming or life-threatening overcrowding (cf. [4]), and how to guide them to escape from the dangerous place has significant importance for social security. Our quantitative results may provide some clues for the number of leaders (e.g., the people or the robots who are familiar with the local situations) needed to guide the people in panic to a safe place.

In this paper, we focus our attention on the quantitative analysis of the proportion of leaders needed for reaching consensus decision making in spatially distributed discrete-time multiagent systems. We start with a basic multiagent model, in which all agents move in the plane with a constant speed, but with heading adjusted according to the information from neighbors defined via the spatial relative distance. The followers update their headings by aligning with the average direction of their neighbors, while the leaders adopt the balance between the average heading of neighbors and the preferred direction. For the spatially distributed MASs with multiple leaders, the neighbors of each agent are determined by the spatial relative positions, the positions are determined by the moving headings, and the heading of each agent is determined by the headings of its neighbors. The coupled relationship makes theoretical analysis quite challenging. We proceed with our analysis under a random framework. In comparison with the system with only one leader, we provide a comprehensive analysis for the information transfer between the subgroup of leaders and the subgroup of followers by relying on the system structure and some estimations for the characteristics, and establish some quantitative results for the proportion of leaders needed for the stable consensus decision making. In particular, we consider two cases: (i) the leaders have the same preference, and (ii) the leaders have two different preferences. For case (i), we present an upper bound and a lower bound for the proportion of leaders needed for the stable consensus decision making. When the proportion is above the lower bound, all agents can make a stable decision. However, the decisions are unstable below the upper bound. In case (ii), the two subgroups of leaders affect the followers simultaneously. The key challenge for the convergence analysis lies in the coupling of the headings of different subgroups of agents. We conduct our analysis by considering the worst case. We show that if the proportion of leaders with majority is larger than a lower bound and the proportion of leaders with minority is lower than an upper bound, then the followers will move with the preferred heading of the leaders with majority, and a stable consensus can be achieved.

The rest of this paper is organized as follows. In section 2, we introduce the multiagent model where the leaders have the same preferred direction or two different preferred directions, and present the main results. The proofs of the main theorems are detailed in sections 3 and 4, respectively. Concluding remarks are presented in section 5.

2. Problem formulation and main results. In this paper, we consider the consensus decision-making problem of spatially distributed discrete-time MASs composed of heterogeneous agents: leaders and followers. All agents move in a plane at a constant speed, with headings updated according to the information they receive. The leaders have the preference or knowledge of the moving direction, and their headings are taken as the balance between the average heading of neighbors and the preferred direction, while the followers have no such a preference, and they update their headings by aligning with the average direction of neighbors. Two agents are neighbors if and only if their distance is less than a predefined sensing radius. As the agents move around, the relative distance between agents may change, and thus the neighbor relations between agents may change over time.

We say that the stable consensus decision making is achieved if all followers move with the same preferred direction eventually. The purpose of the current paper is to quantitatively investigate how the proportion of leaders in the MAS affects the stable consensus decision making for two cases where the leaders have the same preferred direction and the leaders have two conflict preferences.

2.1. Leaders with one preferred direction. Consider a MAS with n agents, where n_1 agents have the information of the preferred direction. In this subsection, we assume that the leaders have the same preference. The proportion of leaders is represented by $\alpha_n = \frac{n_1}{n}$. The whole population is composed of two subgroups of agents: leaders and followers. We denote the leader set and the follower set as $V_1 = \{1, 2, \ldots, n_1\}$ and $V_2 = \{n_1 + 1, n_1 + 2, \ldots, n\}$, respectively, and $V = V_1 \bigcup V_2$.

We let $X_i(t) = (x_i(t), y_i(t))^\tau \in \mathbb{R}^2$ and $\theta_i(t)$ denote the position and the moving heading of an agent i ($i \in V$) at the discrete time t ($t = 0, 1, \ldots$), respectively. All agents move in the plane with a constant speed v_n, and thus the velocity of the agent i is $v_n(\cos\theta_i(t), \sin\theta_i(t))^\tau$. Hence, the positions of both leaders and followers are updated according to the following equation,

$$X_i(t+1) = X_i(t) + v_n(\cos\theta_i(t), \sin\theta_i(t))^\tau. \tag{2.1}$$

The preferred direction of leaders is denoted by $\overline{\theta}_0$. The headings of leaders are adopted as a balance between the steering term $\overline{\theta}_0$ and the alignment term with their local neighbors by a weighting term ω ($0 < \omega \leq 1$). Thus, for a leader i ($i \in V_1$), its heading is updated according to the following equation,

$$\theta_i(t+1) = (1-\omega)\frac{\sum_{j \in N_i(t)} \theta_j(t)}{n_i(t)} + \omega\overline{\theta}_0, \tag{2.2}$$

where $N_i(t)$ denotes the neighbor set of the agent i at time t, and $n_i(t)$ is the cardinality of the set $N_i(t)$. The heading update law of leaders (2.2) is a simplified version of the decision law used in [19], and includes the case where the moving heading of leaders is chosen as the preferred direction directly, while the followers have no preferred directions, and they move with the average heading of their neighbors, i.e., for $i \in V_2$,

$$\theta_i(t+1) = \frac{\sum_{j \in N_i(t)} \theta_j(t)}{n_i(t)}. \tag{2.3}$$

Remark 2.1. The headings of all agents take values in the set $[-\pi, \pi)$, and the heading update (2.3) may sometimes cause some counterintuitive consequences. For example, suppose that an agent has only two neighbors (including itself) with the heading angles $-\pi$ and π, respectively. According to (2.3), the heading at the next time step becomes 0, meaning that the agent needs to reverse its direction even though its current heading is the same as its neighbor. One alternative to avoid such a situation is to restrict the initial headings of all agents in half a circle. It is worth noting that issues along these lines would not arise if the systems under consideration were used to other physical variables such as speed, temperature, or opinion. Therefore, we adopt (2.3) to update the states of the agents without loss of generality.

The neighbor relation is one of the key factors for the evolution of headings of all agents. In this paper, the widely used circular neighborhood for all agents is adopted. That is, for an agent i, its neighbors are those agents lying within the circle centered at agent i's current position with a predefined radius r_n, i.e.,

$$N_i(t) = \{j : d_{ij}(t) < r_n\}, \tag{2.4}$$

where $d_{ij}(t) = \|X_j(t) - X_i(t)\|$. From (2.4), each agent is a neighbor of itself. The neighbors of each agent can be divided into the leader neighbor set $N_{i1}(t)$ and the follower neighbor set $N_{i2}(t)$ which are, respectively, defined as follows:

$$N_{i1}(t) = \{j \in V_1 : \ d_{ij}(t) < r_n\}, \tag{2.5}$$

$$N_{i2}(t) = \{j \in V_2 : \ d_{ij}(t) < r_n\}. \tag{2.6}$$

The cardinalities of the sets $N_{i1}(t)$ and $N_{i2}(t)$, i.e., the leader degree and the follower degree of the agent i at time t, are denoted as $n_{i1}(t)$ and $n_{i2}(t)$, respectively. It is clear that $N_i(t) = N_{i1}(t) \bigcup N_{i2}(t)$, and $n_i(t) = n_{i1}(t) + n_{i2}(t)$.

Remark 2.2. Though the headings of both leaders and followers linearly depend on the headings of their local neighbors at a given time instant, the headings of all agents nonlinearly evolve with time due to the coupled relationship between positions, headings, and neighbors of all agents.

As the number of agents increases, so does the number of neighbors. Thus, the interaction radius can be allowed to decrease with the number of agents n which is in the spirit of the standard results established in either wireless networks (see [24], [25]) or random graphs (see [38], [39]). For such a case, the moving speed should also decrease with n. In this section, our purpose is to establish scaling laws for the proportion of leaders needed for the stable consensus in a sense that the followers move with desired heading $\overline{\theta}_0$ eventually.

From the dynamics of the multiagent model (2.1)–(2.6), we know that the initial distribution of the agents plays an important role for the dynamical property of the system. To illustrate this point, we consider an extreme situation where the leaders and followers are, respectively, distributed in two disjoint regions at the initial time. We assume that the distance between these two regions is large. Thus, the followers have no leader neighbors at the initial time, which indicates that the information of the preferred direction of leaders cannot be transferred to the subgroup of followers, while by (2.1), the leaders will converge to the preferred direction with a certain rate no matter what their neighbors are. As a consequence, the followers cannot track the preferred direction in a short time period due to the large distance between the subgroup of leaders and the subgroup of followers, even if the number of leaders is large. Hence, the distance between the subgroup of leaders and the subgroup of followers may become larger and larger because of the dissimilarity between the headings of leaders and followers. For such a situation, it is impossible for the followers to make a stable consensus decision as t tends to infinity. In order to investigate the role of the proportion of leaders for consensus, we conduct our analysis under the following setting.

Assumption 2.1. (1) The initial positions of all agents are uniformly and independently distributed (u.i.d.) in the unit square $[0,1]^2$.

(2) The initial headings of all agents are u.i.d. in the interval $[-\pi,\pi)$, and the initial positions and headings of all agents are independent.

Remark 2.3. Couzin et al. [19] qualitatively investigated the proportion of leaders needed for the desired behavior under the assumption that the initial positions are u.i.d. (see [37] for the definition). This assumption makes it meaningful and possible to investigate how the proportion of leaders affects the consensus decision making. The uniform assumption for the initial headings can also be replaced by other distributions, under which the dissimilarity between the headings of followers and the preferred heading at the initial time can be calculated using the proof idea of Lemma A.4, and the number of leaders needed for the stable consensus may be obtained accordingly.

From an intuitive point of view, if the number of leaders is very small in comparison with the number of followers, e.g., one single leader, then very few leaders fall within the neighborhood of each follower, and thus the influence of leaders on the followers is weak and the preferred direction is transferred to the subgroup of followers with a low rate. The followers tend to move in a self-organized manner, while the leaders converge to the preferred direction with a fixed rate. As a result, the whole population splits into two disjoint clusters: cluster of leaders and cluster of followers. The following theorem establishes an impossibility theorem (necessary condition) for the proportion of leaders to reach the stable consensus decision.

THEOREM 2.4 (necessity). *Suppose that* $\overline{\theta}_0 \neq 0$*, and that the neighborhood radius and the moving speed satisfy* $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$ *and* $v_n \ll r_n$.[1] *If the proportion of leaders satisfies*

$$\alpha_n \ll v_n r_n \sqrt{\frac{\log n}{n}}, \tag{2.7}$$

then under Assumption 2.1*, the system* (2.1)–(2.6) *cannot make stable consensus decisions almost surely for large* n.

[1]For any two positive sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, $a_n = O(b_n)$ means that there exists a positive constant C independent of n, such that $a_n \leq C b_n$ for any $n \geq 1$; $a_n = o(b_n)$ (or $(a_n \ll b_n)$) means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Remark 2.5. The condition $\overline{\theta}_0 \neq 0$ in Theorem 2.4 cannot be removed. From Lemma A.4 in Appendix A, we see that the headings of followers at time $t = 1$ are close to 0. If $\overline{\theta}_0 = 0$, then the influence of leaders and the self-organized behavior of followers cannot be differentiated, and thus it is impossible to obtain the necessary condition for the stable consensus decisions.

Remark 2.6. In Theorem 2.4, the condition on the neighborhood radius $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$ is used to estimate the initial leader degrees and follower degrees and the properties of headings of all agents at $t = 1$, which is a necessary step to analyze the dynamical behavior of the whole system. As the system evolves, the condition on the moving speed is used to guarantee that the change of the number of neighbors is not too much. Moreover, by Lemma A.4 in Appendix A, we see that at $t = 1$, the headings of leaders are close to $\omega\overline{\theta}_0$ and the headings of followers are close to 0. If the proportion of leaders satisfies (2.7), the influence of leaders on followers is too small to induce the followers to track the preferred heading $\overline{\theta}_0$ even if all leaders are the neighbors of followers. Then all agents will be separated into two clusters after a period of time. Hence, the system cannot reach the stable consensus.

If the proportion of leaders is larger, then it is possible to guide the followers to the preferred direction eventually. The following theorem presents a sufficient condition on the proportion of leaders required for the stable consensus decision making.

THEOREM 2.7 (sufficiency). *Suppose that the neighborhood radius and the moving speed satisfy* $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$ *and* $v_n \ll r_n$. *Then under Assumption* 2.1*, the system* (2.1)–(2.6) *achieves the stable consensus decisions almost surely for large* n *if the proportion of leaders satisfies one of the following two conditions:*

(1) $\alpha_n \gg \frac{\log n}{nr_n^2}$ *provided that* $\frac{\log n}{nr_n} \gg v_n$.

(2) $\omega\alpha_n \geq \frac{2049 v_n(|\overline{\theta}_0|+L_n)}{r_n}$ *with* $L_n = 4\sqrt{\frac{3\pi \log n}{nr_n^2}}(1+o(1))$ *provided that* $\frac{\log n}{nr_n} \ll v_n$.

The difference between Theorem 2.7 and the main results established in [36] lies in two aspects: (i) The main results in [36] are based on the nonlinear Vicsek model, while Theorem 2.7 in this paper is based on the linearized Vicsek model which has a simpler expression and more practical applications in, e.g., coordination control of robots and localization of the distributed satellites. (ii) The heading update equation (2.2) in this paper has a more general expression than that used in [36].

Remark 2.8. In Theorem 2.7, the condition $\alpha_n \gg \frac{\log n}{nr_n^2}$ can guarantee that the number of leader neighbors at the initial time is sufficiently large (see Corollary A.2 in Appendix A) while the condition $\omega\alpha_n \geq \frac{2049 v_n(|\overline{\theta}_0|+L_n)}{r_n}$ is used to guarantee that the leader neighbors do not change too much during the evolution, which makes it possible for the followers to track the preferred direction of leaders.

Remark 2.9. We take $r_n = (\frac{\log n}{n})^{1/2-\delta}$ and $v_n = (\frac{\log n}{n})^{1/2+2\delta}$ with $0 < \delta < 1/2$. By Theorem 2.7, it is clear that if the proportion of leaders satisfies the condition $\alpha_n \gg (\frac{\log n}{n})^{2\delta}$, then the followers can reach the desired behavior eventually. On the contrary, by Theorem 2.4 if the proportion of leaders satisfies $\alpha_n \ll (\frac{\log n}{n})^{3/2+\delta}$, then the followers cannot be guided to the desired heading. The gap as shown in Figure 1 will converge to 0 as n increase to infinity. How all agents behave if the proportion of leaders falls into the gap is an open problem, and we will investigate it in future research.

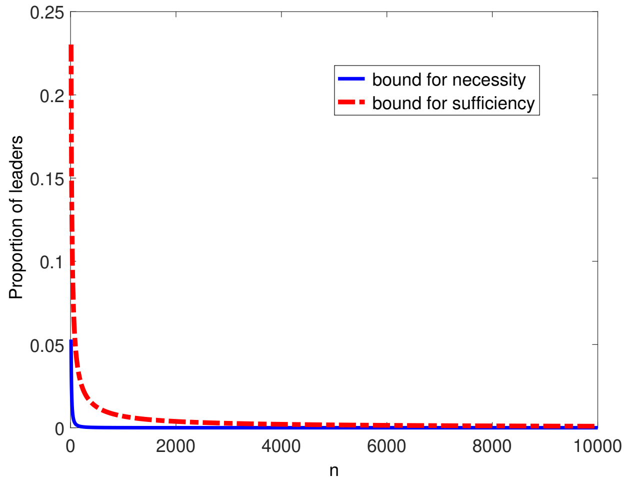

Fig. 1. *The system can reach stable consensus if the proportion of leaders is above the red line, and cannot reach stable consensus if the proportion of leaders is below the blue line.*

2.2. Leaders with two different preferred directions. When the leaders have the same preferred direction, we have investigated the proportion of leaders that is needed for the stable consensus decision making in subsection 2.1. However, the leaders may have different preferences in many systems, such as multiple parties in social opinion dynamics and multiple species in biological systems. The investigation for the MASs with different leaders is more practical but more complex.

In this section, we consider the MAS where the leaders have two conflicting preferences, and establish some quantitative results for the stable consensus. For such a case, the n agents can be classified into three subgroups: two subgroups of leaders denoted as $S1$ and $S2$ and one subgroup of followers denoted as $S3$; each subgroup of leaders has one preferred direction. The preferred direction of leaders in subgroup $S1$ is denoted as $\overline{\theta}_1$, and the preferred direction of leaders in subgroup $S2$ is denoted as $\overline{\theta}_2$ ($\overline{\theta}_2 \neq \overline{\theta}_1$). The agents in subgroup $S3$ have no preference about the destination and they update their headings according to the information from neighbors. We use $V_1 = \{1, 2, \ldots, n_1\}$, $V_2 = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}$, and $V_3 = \{n_1+n_2+1, n_1+n_2+2, \ldots, n\}$ to denote the sets of agents in subgroups $S1, S2$, and $S3$, respectively, and $V = V_1 \bigcup V_2 \bigcup V_3$, where n_1 and n_2 are the numbers of agents in subgroups $S1$ and $S2$, respectively. Denote the proportions of leaders as $\alpha_{1n} = \frac{n_1}{n}$ and $\alpha_{2n} = \frac{n_2}{n}$.

For an agent i ($i \in V$), its position and heading at discrete time t ($t = 0, 1, 2, \ldots$) are also denoted by $X_i(t)$ and $\theta_i(t)$, respectively. All agents move in the plane with a constant speed v_n. The position of the agents is updated according to (2.1). For simplicity of analysis, we assume that the agents in subgroups $S1$ and $S2$ move with the preferred headings $\overline{\theta}_1$ and $\overline{\theta}_2$, respectively, i.e., $\theta_i(t) = \overline{\theta}_1$ for $i \in V_1$, and $\theta_i(t) = \overline{\theta}_2$ for $i \in V_2$, while the followers in subgroup $S3$ update their headings according to the average of their neighbors. That is, their headings are updated according to (2.3), where $N_i(t)$ is the neighbor set of the agent i at time t, and $n_i(t)$ is the cardinality of the set $N_i(t)$. The neighbors of followers are defined in the same way as those in section 2.1, which are composed of the following three parts, $N_{i1}(t), N_{i2}(t), N_{i3}(t)$. The cardinalities of the sets $N_{i1}(t)$, $N_{i2}(t)$, and $N_{i3}(t)$, i.e., the leader degrees in subgroups $S1$ and $S2$ and the follower degree of the agent i at time t, are denoted as

$n_{i1}(t)$, $n_{i2}(t)$ and $n_{i3}(t)$, respectively. It is clear that $N_i(t) = N_{i1}(t) \bigcup N_{i2}(t) \bigcup N_{i3}(t)$, and $n_i(t) = n_{i1}(t) + n_{i2}(t) + n_{i3}(t)$. Thus, the heading update equation of followers can be rewritten as the following expression:

$$\theta_i(t+1) = \frac{\sum_{j \in N_{i1}(t) \bigcup N_{i2}(t) \bigcup N_{i3}(t)} \theta_j(t)}{n_{i1}(t) + n_{i2}(t) + n_{i3}(t)}. \tag{2.8}$$

Similarly to the case where the leaders have the same preference, we proceed with our investigation under the following assumption on the initial states of all agents.

Assumption 2.2. (1) The initial positions of all agents obeys the same distribution as those in Assumption 2.1.

(2) The initial headings of followers are u.i.d. in the interval $[-\pi, \pi)$, and they are independent of the positions of all agents.

Different proportion of leaders in subgroups $S1$ and $S2$ may cause the followers to make different decisions. The purpose of this subsection is to establish a sufficient condition and a necessary condition for the proportions of leaders for the stable consensus decisions making. Here the stable consensus means that the followers move along one of the two preferred directions eventually. We first consider a situation where the proportion of leaders in $S1$ is larger than that in $S2$, and investigate under what conditions the system can achieve the stable consensus decisions. For each follower, the number of its leader neighbors in $S1$ is larger than that in $S2$ because of the uniform distribution of the initial positions. When we consider the convergence of the headings of followers to $\overline{\theta}_1$, the leaders in subgroup $S2$ have negative effect. Hence, a larger proportion of leaders will be needed to achieve stable consensus decisions in comparison with the case with one preferred direction. The following theorem establishes a quantitative result on the proportions of leaders α_{1n} and α_{2n} that enable the followers to make the stable consensus decisions.

THEOREM 2.10. *Assume $\overline{\theta}_2 \neq 0$, and that the neighborhood radius and the speed satisfy $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$ and $v_n \ll r_n$. Then under Assumption 2.2, the followers can be guided to move with the same heading $\overline{\theta}_1$ eventually, if the proportion of leaders in subgroup $S2$ satisfies $\alpha_{2n} \ll \alpha_{1n}^2 r_n^3$, and the proportion of leaders in subgroup $S1$ satisfies one of the following two conditions:*

(1) $\alpha_{1n} \gg \sqrt{\frac{\log n}{nr_n^2}}$ *provided that* $\frac{\log n}{nr_n^2} \gg \frac{v_n}{r_n}$.

(2) $\alpha_{1n} \geq 17\sqrt{\frac{v_n(|\overline{\theta}_0|+f_n)}{r_n}}$ *with* $f_n = 4\sqrt{\frac{3\pi \log n}{nr_n^2}}(1+o(1))$ *provided that* $\frac{\log n}{nr_n^2} \ll \frac{v_n}{r_n}$.

Theorem 2.10 provides a sufficient condition for the stable consensus for the case where the proportion of leaders in subgroup $S1$ is larger than that in $S2$. However, if the proportions of two subgroups of leaders are small, then neither the leaders in subgroup $S1$ nor those in subgroup $S2$ have the capability to guide the followers to make the stable consensus decisions, which will be illustrated in Theorem 2.11.

THEOREM 2.11. *Assume that $\overline{\theta}_1 \neq 0$ and $\overline{\theta}_2 \neq 0$. Suppose that the neighborhood radius and the speed satisfy the conditions used in Theorem 2.10. If $\alpha_{1n} + \alpha_{2n} \ll v_n r_n \sqrt{\frac{\log n}{n}}$, then the followers will move with neither $\overline{\theta}_1$ nor $\overline{\theta}_2$ almost surely.*

The proof of Theorem 2.11 is similar to that of Theorem 2.4, and we omit the proof details.

Remark 2.12. For the system with two subgroups of leaders and one subgroup of followers, the analysis is difficult and the results are complex due to the strongly

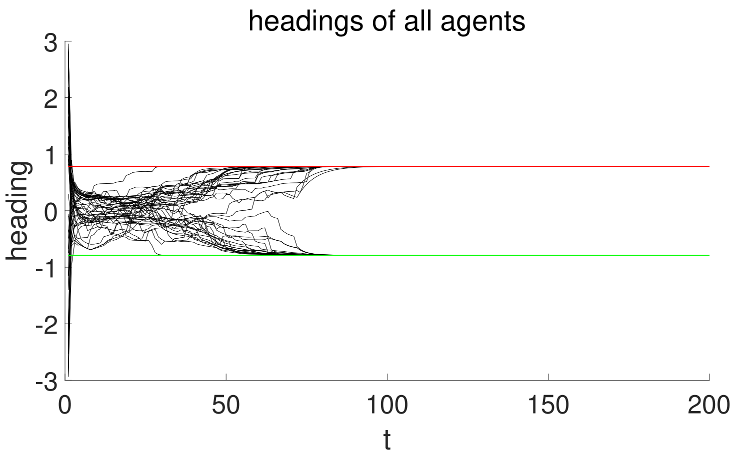

FIG. 2. *The headings of all agents for the system in Example* 2.1.

coupled relationship between positions and headings of all agents and the influence of the two subgroups of leaders to followers. Besides the two cases studied in this paper, there are other possibilities for the behavior of the system, such as some followers follow $\overline{\theta}_1$ and others follow $\overline{\theta}_2$. We illustrate this point by a simulation example.

Example 2.1. Consider a system composed of 50 followers and 20 leaders in subgroup $S1$ and 20 leaders in subgroup $S2$ whose initial states are distributed according to Assumption 2.2, the model parameters are taken as $r = 0.2$ and $v = 0.01$. The headings of the two subgroups of leaders are taken as $\overline{\theta}_1 = \frac{\pi}{4}$, $\overline{\theta}_2 = -\frac{\pi}{4}$. By Figure 2, it is clear that some followers track the desired heading $\overline{\theta}_1$ and others track $\overline{\theta}_2$.

3. Proof of Theorems 2.4 and 2.7.

Under Assumption 2.1, we can obtain the estimation for the initial leader degree and follower degree and the properties of headings at $t = 1$ (see Appendix A), which will be used to prove Theorems 2.4 and 2.7.

From Lemma A.4 in Appendix A, the headings of followers at time $t = 1$ belong to the interval $[-L_n, L_n]$ with $L_n = 4\sqrt{\frac{3\pi \log n}{nr_n^2}}(1 + o(1)) = o(1)$. It is clear that there is a difference between $\theta_i(1)$ $(i \in V_2)$ and the preferred direction $\overline{\theta}_0$ $(\overline{\theta}_0 \neq 0)$. In order to prove Theorem 2.4, we first show that if the proportion of leaders satisfies the conditions in Theorem 2.4, then the followers cannot be guided to the preferred heading $\overline{\theta}_0$ even after a long time period.

LEMMA 3.1. *Suppose that the neighborhood radius and the moving speed satisfy* $\sqrt{\log n/n} \ll r_n \ll 1$ *and* $v_n \ll r_n$. *If the proportion of leaders satisfies the condition* $\alpha_n \ll v_n r_n\sqrt{\frac{\log n}{n}}$, *then for* $0 \le t \le t^* + 1$ *with* $t^* = \lceil \frac{1+2v_n+r_n}{v_n(1-\cos\omega\overline{\theta}_0)} \rceil$, *the headings of followers satisfy*

$$\max_{n_1+1 \le i \le n} |\theta_i(t)| = O\left(\sqrt{\log n/nr_n^2}\right).$$

Proof. Without loss of generality we assume $\overline{\theta}_0 > 0$. By (2.3), the headings of followers are updated according to the headings of their neighbors, where the neighbors are defined via the positions of all agents. In order to investigate the property of the headings of followers, we first prove that for $0 \leq t \leq t^*$, the distance between any two followers i and j satisfies

$$|d_{ij}(t) - d_{ij}(0)| \leq \frac{1}{32} r_n. \tag{3.1}$$

It is clear that (3.1) holds for $t = 0$. We assume that it holds for $0 \leq s \leq t$ $(\leq t^* - 1)$. If the distance between two followers i and j at the initial time satisfies $d_{ij}(0) < (1 - \frac{1}{32})r_n$, then at time s we have $d_{ij}(s) \leq d_{ij}(0) + |d_{ij}(s) - d_{ij}(0)| < r_n$, and if $d_{ij}(0) > (1 + \frac{1}{32})r_n$, then $d_{ij}(s) \geq d_{ij}(0) - |d_{ij}(s) - d_{ij}(0)| \geq r_n$. Thus, for a follower i $(i \in V_2)$, the change of its follower neighbors at time s in comparison with those at the initial time is included in the set $R_i = \{j \in V_2 : \frac{31}{32} r_n \leq d_{ij}(0) \leq \frac{33}{32} r_n\}$. The cardinality of the set R_i is denoted as r_i which is bounded by $r_{\max} \triangleq \max_{i \in V_2} r_i \leq \frac{1}{8} n\pi r_n^2(1 + o(1))$; see Lemma 11 in [36]. Thus, for each follower agent i $(i \in V_2)$, the number of its follower neighbors at time s satisfies $\max_{i \in V_2} |n_{i2}(s) - n_{i2}(0)| \leq r_{\max}$, i.e., $\min_{i \in V} n_{i2}(0) - r_{\max} \leq n_{i2}(s) \leq \max_{i \in V} n_{i2}(0) + r_{\max}$. Moreover, by Corollary A.2, we have

$$\frac{1}{8} n\pi r_n^2(1 + o(1)) \leq n_{i2}(s) \leq \frac{9}{8} n\pi r_n^2(1 + o(1)). \tag{3.2}$$

Using Lemma A.4 and the heading update equations (2.2) and (2.3), we see that the headings of leaders satisfy $\max_{i \in V_1} |\theta_i(s)| \leq \overline{\theta}_0$. Hence, by (2.3), the headings of followers satisfy

$$\begin{aligned}
\max_{i \in V_2} |\theta_i(s+1)| &= \max_{i \in V_2} \left| \frac{\sum_{j \in N_{i1}(s)} \theta_j(s) + \sum_{j \in N_{i2}(s)} \theta_j(s)}{n_{i1}(s) + n_{i2}(s)} \right| \\
&\leq \max_{i \in V_2} \left(\frac{n_{i1}(s)\overline{\theta}_0 + n_{i2}(s) \max_{i \in V_2} |\theta_i(s)|}{n_{i1}(s) + n_{i2}(s)} \right) \\
&\leq \max_{i \in V_2} \frac{n_1 \overline{\theta}_0}{n_1 + n_{i2}(s)} + \max_{i \in V_2} \frac{n_{i2}(s) \max_{i \in V_2} |\theta_i(s)|}{n_1 + n_{i2}(s)},
\end{aligned} \tag{3.3}$$

where $\frac{n_{i1}(s)\overline{\theta}_0 + n_{i2}(s) \max_{i \in V_2} |\theta_i(s)|}{n_{i1}(s) + n_{i2}(s)}$ is monotonically increasing with $n_{i1}(s)$ due to the inequality $\max_{i \in V_2} |\theta_i(s)| \leq \overline{\theta}_0$.

Set $\gamma(s) = \max_{i \in V_2} \frac{n_1}{n_1 + n_{i2}(s)}$ and $\beta(s) = \max_{i \in V_2} \frac{n_{i2}(s)}{n_1 + n_{i2}(s)}$. Then by (3.3), we have

$$\begin{aligned}
\max_{i \in V_2} |\theta_i(s+1)| &\leq \gamma(s)\overline{\theta}_0 + \beta(s) \max_{i \in V_2} |\theta_i(s)| \\
&\leq \prod_{k=1}^{s} \beta(k) \max_{i \in V_2} |\theta_i(1)| + \sum_{k=1}^{s} \Big(\prod_{l=k+1}^{s} \beta(l) \cdot \gamma(k)\overline{\theta}_0 \Big) \\
&\leq 4\sqrt{\frac{3\pi \log n}{n r_n^2}}(1 + o(1)) + \sum_{k=1}^{s} \Big(\prod_{l=k+1}^{s} \beta(l) \cdot \gamma(k)\overline{\theta}_0 \Big).
\end{aligned} \tag{3.4}$$

Denote $n_{\max}(k) = \max_{i \in V_2} n_{i2}(k), n_{\min}(k) = \min_{i \in V_2} n_{i2}(k)$. By (2.7), (3.2), and

Corollary A.2, we have

$$
\begin{aligned}
\sum_{k=1}^{s}\Big(\prod_{l=k+1}^{s}\beta(l)\cdot\gamma(k)\Big) &= \sum_{k=1}^{s}\left[\prod_{l=k+1}^{s}\max_{i\in V_2}\frac{n_{i2}(l)}{n_1+n_{i2}(l)}\cdot\max_{i\in V_2}\frac{n_1}{n_1+n_{i2}(k)}\right]\\
&\le \sum_{k=1}^{s}\left[\prod_{l=k+1}^{s}\left(1-\frac{n_1}{n_1+n_{\max}(l)}\right)\cdot\frac{n_1}{n_1+n_{\min}(k)}\right]\\
&\le \frac{n_1}{n_{\min}(0)-r_{\max}+n_1}\sum_{k=1}^{s}\left(\frac{n_{\max}(0)+r_{\max}}{n_{\max}(0)+r_{\max}+n_1}\right)^{s-k}\\
&\le \frac{n_{\max}(0)+r_{\max}+n_1}{n_{\min}(0)-r_{\max}+n_1}\left[1-\left(\frac{n_{\max}(0)+r_{\max}}{n_{\max}(0)+r_{\max}+n_1}\right)^{s}\right]\\
&\le 10\left[1-\left(1-\frac{8n_1(1+o(1))}{9n\pi r_n^2}\right)^{t^*}\right]. \qquad (3.5)
\end{aligned}
$$

Set $\zeta_n=\frac{8n_1}{9n\pi r_n^2}(1+o(1))$. Using $\alpha_n \ll v_n r_n\sqrt{\frac{\log n}{n}}$, we have

$$
t^*\zeta_n=\frac{1+2v_n+r_n}{v_n(1-\cos\omega\overline{\theta}_0)}\frac{8\alpha_n}{9\pi r_n^2}(1+o(1))=o(\sqrt{\log n/(nr_n^2)}).
$$

Thus,

$$
(1-\zeta_n)^{t^*}=1-t^*\zeta_n+O\left((t^*\zeta_n)^2\right)=1-t^*\zeta_n(1+o(1)). \qquad (3.6)
$$

Substituting (3.5) and (3.6) into (3.4), we have

$$
\begin{aligned}
\max_{i\in V_2}|\theta_i(s+1)| &\le 4\sqrt{\frac{3\pi\log n}{nr_n^2}}(1+o(1))+10t^*\zeta_n\overline{\theta}_0(1+o(1))\\
&= 4\sqrt{\frac{3\pi\log n}{nr_n^2}}(1+o(1)). \qquad (3.7)
\end{aligned}
$$

By (2.1), the distance between any two followers i and j satisfies the following inequality for $t\le t^*$:

$$
\begin{aligned}
&|d_{ij}(t+1)-d_{ij}(0)| \le \sum_{s=0}^{t}\big|\|X_i(s+1)-X_j(s+1)\|-\|X_i(s)-X_j(s)\|\big|\\
&\le 2v_n+v_n\sum_{s=1}^{t}\max_{i,j\in V_2}|\theta_i(s)-\theta_j(s)|\le 2v_n+2v_n\sum_{s=1}^{t}\max_{i\in V_2}|\theta_i(s)|\\
&\le 2v_n+8v_nt^*\sqrt{\frac{3\pi\log n}{nr_n^2}}(1+o(1))\\
&\le \frac{1}{32}r_n. \qquad (3.8)
\end{aligned}
$$

Similarly to the analysis of (3.7), we can deduce the result of the lemma for $t\le t^*$. ∎

Proof of Theorem 2.4. Using Lemma A.4, the heading of the leader i ($i\in V_1$) satisfies $\theta_i(1)\ge\omega\overline{\theta}_0-(1-w)L_n>0$, and the heading of the follower i ($i\in V_2$)

satisfies $\theta_i(1) \geq -L_n$, where $L_n = 4\sqrt{\frac{3\pi \log n}{nr_n^2}}(1+o(1)) = o(1)$ for large n. By (2.2) and (2.3), we have for i $(i \in V)$ and $t \geq 1$, $\theta_i(t) \geq \min_{i\in V} \theta_i(1) \geq -L_n$. Hence, the heading of the leader i $(i \in V_1)$ satisfies

$$\theta_i(t) \geq \omega\overline{\theta}_0 - (1-\omega)L_n, \tag{3.9}$$

while by Lemma 3.1, the heading of the follower i $(i \in V_2)$ satisfies $\max_{i\in V_2} |\theta_i(t)| = O(\sqrt{\frac{\log n}{nr_n^2}}) = o(1)$ for $1 \leq t \leq t^* + 1$.

Using the position update (2.1), we have for i $(i \in V)$

$$\begin{cases} x_i(t+1) = x_i(0) + v_n \sum_{s=0}^{t} \cos\theta_i(s), \\ y_i(t+1) = y_i(0) + v_n \sum_{s=0}^{t} \sin\theta_i(s). \end{cases}$$

Thus, for a leader i $(i \in V_1)$ and a follower j $(j \in V_2)$

$$\begin{aligned} d_{ij}(t^*+1) &\geq |x_j(t^*+1) - x_i(t^*+1)| \\ &= \Big|(x_j(0) - x_i(0)) + v_n \sum_{s=0}^{t^*} (\cos\theta_j(s) - \cos\theta_i(s))\Big| \\ &\geq v_n\Big|\sum_{s=1}^{t^*} (\cos\theta_j(s) - \cos\theta_i(s))\Big| - 2v_n - |(x_i(0) - x_j(0))| \\ &\geq v_n t^*(1 - \cos\omega\overline{\theta}_0)(1+o(1)) - 2v_n - 1 \\ &> r_n. \end{aligned} \tag{3.10}$$

By (3.10), it is clear that the distance between any leader i $(i \in V_1)$ and follower j $(j \in V_2)$ at time t^*+1 is larger than the predefined interaction radius r_n. Thus, the leaders will have no follower neighbors, and the followers will have no leader neighbors. Hence, the followers will not be affected by the reference signal $\overline{\theta}_0$. By Lemma 3.1 and the heading update equation (2.3), the headings of followers satisfy

$$\max_{i\in V_2} |\theta_i(t^*+1)| = O(\sqrt{\log n/(nr_n^2)}). \tag{3.11}$$

Repeating the above process, it can be concluded that (3.10) and (3.11) hold for all $t \geq t^* + 1$, which means that the followers will not move with $\overline{\theta}_0$ eventually. ∎

Now, we move on to the proof of Theorem 2.7. Denote $\varepsilon_i(t) = \theta_i(t) - \overline{\theta}_0$. In the investigation of the stable consensus decision making, what is important is the convergence of $\varepsilon_i(t)$. We first provide a preliminary result for the convergence of $\varepsilon_i(t)$ in the following lemma.

Lemma 3.2 (see [40]). *If there exist positive constants A and ϵ, such that the following inequalities hold,*

$$\max_{i\in V_1} |\varepsilon_i(1)| \leq (1-\omega)A, \quad \max_{i\in V_2} |\varepsilon_i(1)| \leq A, \tag{3.12}$$

$$\max_{i\in V} |\alpha_i(t) - \alpha_i(0)| \leq \epsilon, \quad \forall t \geq 1, \tag{3.13}$$

where $\alpha_i(t) = \frac{n_{i1}(t)}{n_{i1}(t)+n_{i2}(t)}$, then we have for $t \geq 0$,

$$\max_{i\in V_1} |\varepsilon_i(t+1)| \leq (1-\omega)\gamma^t A, \quad \max_{i\in V_2} |\varepsilon_i(t+1)| \leq \gamma^t A,$$

where $\gamma = \max_{i\in V}(1 - (\alpha_i(0) - \epsilon)\omega)$.

The proof of Lemma 3.2 can be referred to [40], and we omit it to save space. From Lemma 3.2, if $1-(\alpha_i(0)-\epsilon)\omega<1$, then the convergence of $\max_{i\in V}|\varepsilon_i(t+1)|$ can be achieved. The key point in the proof of Theorem 2.7 is to estimate the upper bound of $\max_{i\in V}|\alpha_i(t)-\alpha_i(0)|$.

Proof of Theorem 2.7. We prove for any agent i $(i\in V)$

$$\max_{i\in V}|\alpha_i(s)-\alpha_i(0)|<\frac{\alpha_n}{2}, \tag{3.14}$$

where $\alpha_i(t)$ is defined in Lemma 3.2. It is clear that the inequality (3.14) holds for $s=0$. We assume that (3.14) holds for $0\le s\le t$, and prove the inequality for $t+1$ by mathematical induction.

Using Lemma A.4 in Appendix A, the headings of leaders and followers at $t=1$ satisfy $\max_{i\in V_1}|\varepsilon_i(1)|\le(1-\omega)(\overline{\theta}_0+L_n)$ and $\max_{i\in V_2}|\varepsilon_i(1)|\le(\overline{\theta}_0+L_n)$, where $L_n=4\sqrt{\frac{3\pi\log n}{nr_n^2}}(1+o(1))$. Using Lemma 3.2, we have for $0\le s\le t$

$$\max_{i\in V_1}|\varepsilon_i(s+1)|\le(1-\omega)\left(1-\frac{\omega\alpha_n}{2}(1+o(1))\right)^s|(\overline{\theta}_0|+L_n), \tag{3.15}$$

$$\max_{i\in V_2}|\varepsilon_i(s+1)|\le\left(1-\frac{\omega\alpha_n}{2}(1+o(1))\right)^s|(\overline{\theta}_0|+L_n). \tag{3.16}$$

By (3.15) and (3.16), the distance between agents i and j $(i,j\in V)$ satisfies

$$\begin{aligned}|d_{ij}(t+1)-d_{ij}(0)|&\le 2v_n+2v_n\sum_{s=1}^{t}\max_{i\in V}|\varepsilon_i(s)|\\ &\le 2v_n+\frac{4v_n}{\omega\alpha_n}(|\overline{\theta}_0|+L_n)\le\frac{r_n}{512},\end{aligned} \tag{3.17}$$

where the condition $\alpha_n\ge\frac{2049v_n(|\overline{\theta}_0|+L_n)}{\omega r_n}$ is used in the last inequality.

By (3.17), the change of leader neighbors and the change of follower neighbors of the agent i are characterized by $R_{i1}=\{j\in V_1:(1-\frac{1}{512})r_n\le d_{ij}(0)\le(1+\frac{1}{512})r_n\}$ and $R_{i2}=\{j\in V_2:(1-\frac{1}{512})r_n\le d_{ij}(0)\le(1+\frac{1}{512})r_n\}$, respectively. The cardinalities of the sets R_{i1} and R_{i2} are denoted as r_{i1} and r_{i2}. Using the condition $\alpha_n\gg\frac{\log n}{nr_n^2}$, we have $\max_{i\in V}r_{i1}\le\frac{1}{128}n\alpha_n\pi r_n^2(1+o(1))$ and $\max_{i\in V}r_{i2}\le\frac{1}{128}n\pi r_n^2(1+o(1))$; see Lemma 11 in [36]. Thus, the leader degree and follower degree of the agents satisfy $|n_{i1}(t+1)-n_{i1}(0)|\le\max_{i\in V}r_{i1}$ and $|n_{i2}(t+1)-n_{i2}(0)|\le\max_{i\in V}r_{i2}$. By Lemma A.2, we have

$$\begin{aligned}\max_{i\in V}|\alpha_i(t+1)-\alpha_i(0)|&=\max_{i\in V}\left|\frac{n_{i1}(t+1)}{n_{i1}(t+1)+n_{i2}(t+1)}-\frac{n_{i1}(0)}{n_{i1}(0)+n_{i2}(0)}\right|\\ &\le\frac{\left(\max_{i\in V}n_{i1}(0)r_{i2}+\max_{i\in V}n_{i2}(0)r_{i1}\right)}{\min_{i\in V}\{(n_{i1}(t+1)+n_{i2}(t+1))(n_{i1}(0)+n_{i2}(0))\}}\\ &\le\frac{\frac{1}{64}\alpha_n(n\pi r_n^2)^2}{\frac{n\pi r_n^2}{16}(1-\frac{1}{32})n\pi r_n^2}(1+o(1))\le\frac{8}{31}\alpha_n(1+o(1))<\frac{\alpha_n}{2}.\end{aligned}$$

By mathematical induction it is clear that (3.14) holds almost surely for all $t\ge 0$.

Using Lemmas 3.2 and A.4, we have

$$\max_{i\in V}|\varepsilon_i(t)| \to 0 \quad \text{as } t\to\infty.$$

This completes the proof of Theorem 2.7. ∎

4. Proof of Theorem 2.10. Denote $\widetilde{\theta}_i(t)=\theta_i(t)-\overline{\theta}_1$ as the maximum dissimilarity between the heading of the follower i ($i\in V_3$) and the preferred direction $\overline{\theta}_1$ of the subgroup $S1$. We aim to prove that $\widetilde{\theta}_i(t)\to 0$ as $t\to\infty$. In the convergence analysis of $\widetilde{\theta}_i(t)$, the influence of leaders in subgroup $S2$ is negative, and we take the effect of the preferred direction of leaders in subgroup $S2$ as noise. We investigate the evolution of the system from the worst case in the sense that all leaders in subgroup $S2$ fall into the neighborhood of each follower. We show that even if for the worst case, there still exists a time instant T^* such that the headings of the followers at time t ($t\geq T^*$) fall in the open ball centered at $\overline{\theta}_1$ with radius $\frac{\overline{\theta}_1-\overline{\theta}_2}{4}$. That is, the headings of the followers will approach to the preferred heading $\overline{\theta}_1$. In order to make our idea clearly, we first consider the situation where the neighbor graphs keep unchanged.

LEMMA 4.1. *Assume that $\overline{\theta}_2\neq 0$, and the neighbor graphs keep unchanged. Suppose that the neighborhood radius and the moving speed satisfy $\sqrt{\frac{\log n}{n}}\ll r_n\ll 1$ and $v_n\ll r_n$. If the proportion of leaders in subgroups $S1$ and $S2$ satisfy the conditions in Theorem 2.10, then there exists T^*, such that the headings of followers have the following tracking property for $t>T^*$:*

$$\max_{i\in V_3}|\widetilde{\theta}_i(t+1)|\leq \frac{|\overline{\theta}_1-\overline{\theta}_2|}{4}.$$

Proof. Since the neighbor graphs keep unchanged, the degrees of followers are constant and equal to their initial degrees. For simplicity of expression, we denote $n_{i1}(t)$, $n_{i2}(t)$ and $n_{i3}(t)$ as n_{i1}, n_{i2}, and n_{i3}, respectively, and denote $\mathcal{N}_{i3}(t)$ as $\mathcal{N}_{i3}$, and $n_i=n_{i1}+n_{i2}+n_{i3}$. By the heading update equation (2.8), the headings of followers satisfy

$$\begin{aligned}
\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| &= \max_{i\in V_3}\left|\frac{n_{i2}(\overline{\theta}_2-\overline{\theta}_1)+\sum_{j\in\mathcal{N}_{i3}}(\theta_j(t)-\overline{\theta}_1)}{n_i}\right| \\
&\leq \max_{i\in V_3}\frac{n_{i3}}{n_i}\max_{i\in V_3}|\widetilde{\theta}_i(t)|+\max_{i\in V_3}\frac{n_{i2}|\overline{\theta}_2-\overline{\theta}_1|}{n_i} \\
&\triangleq \mu_n\max_{i\in V_3}|\widetilde{\theta}_i(t)|+\nu_n\leq \mu_n^t\max_{i\in V_3}|\widetilde{\theta}_i(1)|+(1+\mu_n+\mu_n^2+\cdots\mu_n^{t-1})\nu_n \\
&\leq \mu_n^t\max_{i\in V_3}|\widetilde{\theta}_i(1)|+\frac{1}{1-\mu_n}\nu_n, \qquad (4.1)
\end{aligned}$$

where $\mu_n=\max_{i\in V_3}\frac{n_{i3}}{n_i}$, and $\nu_n=\max_{i\in V_3}\frac{n_{i2}}{n_i}|\overline{\theta}_2-\overline{\theta}_1|$.

Now, we estimate μ_n and ν_n. First, we need to calculate n_{i3} and n_i. By Lemma B.1 and Remark B.2 in Appendix B, we have for the agent i ($i\in V_3$), $n_{i3}\leq$

$n(1-\alpha_{1n}-\alpha_{2n})A_i + C_3\sqrt{n\pi r_n^2 \log n} = nA_i(1-\alpha_{1n}-\alpha_{2n}+\frac{C_3\sqrt{\pi r_n^2\log n}}{A_i\sqrt{n}})$, where C_3 is a positive constant independent of n, and $\frac{1}{4}\pi r_n^2 \le A_i \le \pi r_n^2$. By the condition $\alpha_{1n} \gg \sqrt{\frac{\log n}{n r_n^2}}$, we have $\frac{C_3\sqrt{\pi r_n^2\log n}}{A_i\sqrt{n}} \le 4C_3\sqrt{\frac{\log n}{n\pi r_n^2}} = o(\alpha_{1n})$. Thus, $n_{i3}(0) \le n(1-\alpha_{1n}(1+o(1)))A_i$. For the total number of neighbors $n_i(0)$, by Lemma B.1, we have $n_i(0) \ge nA_i - C_1\sqrt{n\pi r_n^2\log n} = nA_i(1-\frac{C_1\sqrt{\pi r_n^2 \log n}}{A_i\sqrt{n}}) = nA_i(1+o(\alpha_{1n})) \ge \frac{n\pi r_n^2}{4}(1+o(\alpha_{1n}))$. By the above analysis, we have

$$\mu_n = \max_{i\in V_3}\frac{n_{i3}}{n_i} \le \max_{i\in V_3}\frac{n(1-\alpha_{1n}(1+o(1)))A_i}{nA_i(1+o(\alpha_{1n}))} = 1-\alpha_{1n}(1+o(1)),$$
$$\nu_n = \max_{i\in V_3}\frac{n_{i2}}{n_i}|\overline{\theta}_2-\overline{\theta}_1| \le \frac{4\alpha_{2n}}{\pi r_{n^2}}|\overline{\theta}_2-\overline{\theta}_1|(1+o(1)).$$

Substituting the above two inequalities into (4.1), we obtain

$$\begin{aligned}\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| &\le (1-\alpha_{1n}(1+o(1)))^t(\overline{\theta}_1+f_n)+\frac{4\alpha_{2n}}{\alpha_{1n}\pi r_n^2}|\overline{\theta}_1-\overline{\theta}_2|(1+o(1))\\ &\le (1-\alpha_{1n}(1+o(1)))^t(|\overline{\theta}_1|+f_n)+\frac{1}{6}|\overline{\theta}_1-\overline{\theta}_2|.\end{aligned}$$

Taking

$$T^* = \max\Big\{1, \Big\lceil \log\frac{\frac{1}{12}||\overline{\theta}_1|-\overline{\theta}_2|}{\overline{\theta}_1+f_n}\cdot\frac{1}{\log(1-\alpha_{1n}(1+o(1)))}\Big\rceil\Big\},$$

we have for $t \ge T^*$, $\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| \le \frac{|\overline{\theta}_1-\overline{\theta}_2|}{4}$. ◻

When the neighbor graphs keep unchanged, we have shown that the leaders in subgroup $S1$ can guide the headings of followers close to the preferred heading. For the system under consideration, the neighbor graphs are determined by the positions of all agents, and they may change dynamically with time. In the following, we show that a similar result to that of Lemma 4.1 still holds for the dynamical neighbors graphs.

A clue from Lemma 4.1 is that each follower is required to have enough leader neighbors in subgroup $S1$ to guide the followers to approach the preferred direction $\overline{\theta}_1$, because a too small number of leader neighbors in $S1$ does not have the ability to overcome the "interference" of leaders in $S2$. In Lemma 4.1, the number of the initial leader neighbors in subgroup $S1$ is large enough to guarantee the domination of $\overline{\theta}_1$. In order to maintain this domination, we need to show that the change of the dynamical neighbor graphs is not too much for the case where the neighbor graphs are determined by the distance between agents. In the following analysis, we assume that $0 \le \overline{\theta}_1 \le \frac{\pi}{2}$ for simplicity of expression.

LEMMA 4.2. *For the dynamical neighbor graphs defined via the neighbor relations* (2.6)*, under the conditions of Lemma* 4.1*, the distance between any two agents $i \in V_1\bigcup V_3$ and $j \in V_1\bigcup V_3$ satisfies ($t\in[0,T^*]$),*

$$|d_{ij}(t)-d_{ij}(0)| \le \frac{\alpha_{1n}}{64}r_n, \tag{4.2}$$

where $T^* = \frac{C}{v_n}$ with C being a positive constant independent of n.

Proof. It is clear that (4.2) holds for $t=0$. We assume that it holds for $1 \le t \le t_0$ ($\le T^*-1$), and prove this inequality for $t = t_0+1$. By (2.1), the distance between any two agents $i \in V_1 \bigcup V_3$ and $j \in V_1 \bigcup V_3$ satisfies $|d_{ij}(t_0+1) - d_{ij}(0)| \leq 2v_n + 2v_n \sum_{s=1}^{t_0} \max_{i \in V_3} |\widetilde{\theta}_i(s)|$. In order to estimate $d_{ij}(t_0+1)$, we need to analyze how the maximum dissimilarity $\max_{i\in V_3} |\widetilde{\theta}_i(s)|$ evolves with time. By (2.8), the headings of followers satisfy

$$\begin{aligned}\max_{i\in V_3} |\widetilde{\theta}_i(t+1)| &\leq \max_{i\in V_3} \frac{n_{i3}(t)}{n_i(t)} \max_{i\in V_3} |\widetilde{\theta}_i(t)| + \max_{i\in V_3} \frac{n_{i2}(t)}{n_i(t)} |\overline{\theta}_2 - \overline{\theta}_1| \\ &\triangleq \mu_n(t) \max_{i\in V_3} |\widetilde{\theta}_i(t)| + \nu_n(t) \\ &\leq \Pi_{s=1}^t \mu_n(s) \max_{i\in V_3} |\widetilde{\theta}_i(1)| + \sum_{s=1}^t \Big(\Pi_{k=s+1}^t \mu_n(k)\Big)\nu_n(s), \end{aligned} \tag{4.3}$$

where $\mu_n(t) = \max_{i\in V_3} \frac{n_{i3}(t)}{n_i(t)}$ and $\nu_n(t) = \max_{i\in V_3} \frac{n_{i2}(t)}{n_i(t)} |\overline{\theta}_2 - \overline{\theta}_1|$. For each follower i $(i \in V_3)$, we introduce the following two sets,

$$\mathcal{C}_i = \Big\{ j \in V : \big(1 - \frac{\alpha_{1n}}{64}\big) r_n \leq d_{ij}(0) \leq \big(1 + \frac{\alpha_{1n}}{64}\big) r_n \Big\}, \tag{4.4}$$

$$\mathcal{C}_{i3} = \Big\{ j \in V_3 : \big(1 - \frac{\alpha_{1n}}{64}\big) r_n \leq d_{ij}(0) \leq \big(1 + \frac{\alpha_{1n}}{64}\big) r_n \Big\}. \tag{4.5}$$

Denote c_i and c_{i3} as the number of agents in the sets $\mathcal{C}_i$ and $\mathcal{C}_{i3}$, which are estimated in Lemma B.4 in Appendix B. Because the inequality (4.2) holds for $0 \leq t \leq t_0$, for each follower i $(i \in V_3)$, the total neighbors and follower neighbors changed at time t $(\in [0, t_0])$ in comparison with the corresponding total neighbors and follower neighbors at the initial time are included in the sets $\mathcal{C}_i$ and $\mathcal{C}_{i3}$, respectively. Note that $\max_{i\in V_3} E\{I(j \in \mathcal{C}_{i3})|X_i(0)\} \leq 4\pi \frac{\alpha_{1n}}{64} r_n^2 = \frac{\pi\alpha_{1n} r_n^2}{16}$, where $E\{\cdot|\cdot\}$ denotes the conditional expection. Thus, by Lemma B.4, we have $\max_{i\in V_3} c_{i3} \leq \frac{n(1-\alpha_{1n}-\alpha_{2n})\pi\alpha_{1n} r_n^2}{16} \cdot (1+o(1))$. Similarly, the cardinality of the set $\mathcal{C}_i$ satisfies $\max_{i\in V_3} c_i \leq \frac{n\pi\alpha_{1n} r_n^2}{16}(1 + o(1))$. For the followers, the number of its follower neighbors at time t $(\in [0, t_0])$ satisfies $\max_{i\in V_3} n_{i3}(t) \leq \max_{i\in V_3} n_{i3}(0) + \max_{i\in V_3} c_{i3} \leq n(1-\alpha_{1n}(1+o(1)))A_i + \frac{n(1-\alpha_{1n}-\alpha_{2n})\pi\alpha_{1n} r_n^2}{16}(1+o(1)) \leq nA_i[1-\alpha_{1n}(1+o(1)) + \frac{\pi\alpha_{1n} r_n^2}{16A_i}] \leq nA_i(1 - \frac{3\alpha_{1n}}{4}(1 + o(1)))$. The number of its total neighbors satisfies $n_i(t) \geq \min_{i\in V_3} n_i(0) - \max_{i\in V_3} c_i \geq nA_i(1+o(\alpha_{1n})) - \frac{n\pi\alpha_{1n} r_n^2}{16}(1+o(1)) \geq nA_i[1+o(\alpha_{1n}) - \frac{\pi\alpha_{1n} r_n^2}{16A_i}(1+o(1))] \geq nA_i(1 - \frac{\alpha_{1n}}{4}(1+o(1)))$. Hence, we have

$$\begin{aligned}\mu_n(t) = \max_{i\in V_3} \frac{n_{i3}(t)}{n_i(t)} &\leq \max_{i\in V_3} \frac{nA_i(1-\frac{3\alpha_{1n}}{4}(1+o(1)))}{nA_i(1-\frac{\alpha_{1n}}{4}(1+(1)))} \\ &\leq 1 - \frac{\alpha_{1n}}{2}(1+o(1)) \triangleq \widetilde{\alpha}_{1n}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\nu_n(t) = \max_{i\in V_3} \frac{n_{i2}(t)}{n_i(t)} |\overline{\theta}_2 - \overline{\theta}_1| &\leq \frac{n\alpha_{2n}|\overline{\theta}_2 - \overline{\theta}_1|}{nA_i}(1+o(1)) \\ &\leq \frac{4\alpha_{2n}|\overline{\theta}_2 - \overline{\theta}_1|}{\pi r_n^2}(1+o(1)) \triangleq \widetilde{\nu}_n.\end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.3), we have

$$\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| \le (\widetilde{\alpha}_{1n})^t \max_{i\in V_3}|\widetilde{\theta}_i(1)| + \sum_{s=1}^{t}(\widetilde{\alpha}_{1n})^{t-s}\widetilde{\nu}_n.$$

Thus, we have

$$\begin{aligned}
&|d_{ij}(t_0+1) - d_{ij}(0)| \le 2v_n + 2v_n \sum_{s=1}^{t_0} \max_{i\in V_3}|\widetilde{\theta}_i(s)| \le 2v_n \\
&\quad + 2v_n \left\{ \sum_{s=1}^{t_0} (\widetilde{\alpha}_{1n})^{s-1} \max_{i\in V_3}|\widetilde{\theta}_i(1)| + \left[1 + (1+\widetilde{\alpha}_{1n}) + \cdots + (1+\widetilde{\alpha}_{1n} + \widetilde{\alpha}_{1n}^{t_0-2})\right]\widetilde{\nu}_n \right\} \\
&\le 2v_n \left\{ 1 + \frac{1}{1-\widetilde{\alpha}_{1n}}(|\overline{\theta}_1| + f_n) + T_2^* \frac{1}{1-\widetilde{\alpha}_{1n}}\widetilde{\nu}_n \right\} \\
&\le 2v_n \left(1 + \frac{2}{\alpha_{1n}}(|\overline{\theta}_1| + f_n) + \frac{2C}{v_n\alpha_{1n}} \frac{4\alpha_{2n}}{\pi r_n^2}|\overline{\theta}_2 - \overline{\theta}_1| \right) \\
&\le \frac{4v_n}{\alpha_{1n}}(|\overline{\theta}_1| + f_n) + o(\alpha_{1n}r_n) \le \frac{\alpha_{1n}r_n}{64},
\end{aligned}$$

where the conditions on α_{1n} and α_{2n} are used. ∎

From Lemma 4.2, we see that for the follower i, its total neighbors and follower neighbors changed at time t ($t \le T^*$) are, respectively, included in the sets $\mathcal{C}_i$ and $\mathcal{C}_{i3}$ defined by (4.4) and (4.5). Using this, a result similar to that of Lemma 4.1 is established for the MAS where the neighbor graphs dynamically change with the distance between agents.

LEMMA 4.3. *Under the conditions of Lemma* 4.1*, there exists a time* T_1^* *such that for* $t \in [T_1^*+1, T^*]$ *with* T^* *defined in Lemma* 4.2*,*

$$\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| \le \frac{|\overline{\theta}_1 - \overline{\theta}_2|}{4}.$$

Proof. By Lemma 4.2, we have $\mu_n(t) \le 1 - \frac{\alpha_{1n}}{2}(1+o(1)) \triangleq \widetilde{\alpha}_{1n}$ and $\nu_n(t) \le \frac{4\alpha_{2n}|\overline{\theta}_2 - \overline{\theta}_1|}{\pi r_n^2}(1+o(1))$. Using (4.3), the headings of followers satisfy the following relation:

$$\begin{aligned}
&\max_{i\in V_3}|\widetilde{\theta}_i(t+1)| \\
&\le (\widetilde{\alpha}_{1n})^t(|\overline{\theta}_1| + f_n) + (1 + \widetilde{\alpha}_{1n} + \widetilde{\alpha}_{1n}^2 + \cdots + \widetilde{\alpha}_{1n}^{t-1})\frac{4\alpha_{2n}|\overline{\theta}_2 - \overline{\theta}_1|}{\pi r_n^2}(1+o(1)) \\
&\le (\widetilde{\alpha}_{1n})^t(|\overline{\theta}_1| + f_n) + \frac{8\alpha_{2n}|\overline{\theta}_2 - \overline{\theta}_1|}{\alpha_{1n}\pi r_n^2}(1+o(1)) \\
&\le (\widetilde{\alpha}_{1n})^t(|\overline{\theta}_1| + f_n) + \frac{|\overline{\theta}_2 - \overline{\theta}_1|}{6},
\end{aligned}$$

where the last inequality holds due to $\alpha_{1n}r_n^2 \gg r_n\sqrt{\frac{\log n}{n}} \gg \alpha_{2n}$. By taking

$$T_1^* = \max\left\{1, \left\lceil \log \frac{\frac{1}{12}|\overline{\theta}_1 - \overline{\theta}_2|}{|\overline{\theta}_1| + f_n} \cdot \frac{1}{\log(\widetilde{\alpha}_{1n})}\right\rceil\right\},$$

we have for $t \in [T_1^*+1, T^*]$, $\max_{i\in V_3} |\widetilde{\theta}_i(t+1)| \leq \frac{|\overline{\theta}_2 - \overline{\theta}_1|}{4}$. ∎

From Lemma 4.3, the dissimilarity between the headings of agents in subgroup $S3$ and the preferred direction $\overline{\theta}_2$ for $t \in [T_1^*+1, T^*]$ satisfies $\max_{i\in V_3} |\theta_i(t) - \overline{\theta}_2| \geq |\overline{\theta}_1 - \overline{\theta}_2| - \max_{i\in V_3} |\theta_i(t) - \overline{\theta}_1| \geq \frac{3|\overline{\theta}_1 - \overline{\theta}_2|}{4}$. If the length of the time interval $[T_1^*+1, T^*]$ is large enough, then the distance between the agents in subgroup $S2$ and the agents in subgroup $S3$ is larger than r_n, and they will not be neighbors.

LEMMA 4.4. *Under the conditions of Theorem* 2.10*, the distance between the leader i ($i \in V_2$) in subgroup $S2$ and the follower j ($j \in V_3$) for $t \geq T^*$ with T^* defined in Lemma* 4.2 *satisfies*

$$d_{ij}(t) \geq r_n.$$

Proof. We will prove the lemma by considering two cases.

Case I: $0 < \overline{\theta}_2 \leq \pi$. Note that $0 \leq \overline{\theta}_1 \leq \frac{\pi}{2}$. By Lemma 4.3, for $j \in V_3$, if $\overline{\theta}_2 < \overline{\theta}_1$, then for $t \in [T_1^*+1, T^*]$, $\pi \geq \theta_j(t) \geq \overline{\theta}_1 - \frac{\overline{\theta}_1 - \overline{\theta}_2}{4} \geq \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} > \overline{\theta}_2 \geq 0$; and if $\overline{\theta}_2 > \overline{\theta}_1$, then $0 \leq \theta_j(t) \leq \overline{\theta}_1 + \frac{\overline{\theta}_2 - \overline{\theta}_1}{4} = \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} < \overline{\theta}_2 \leq \pi$. Taking $T_2^* = \lceil \frac{1 + r_n + 2v_nT_1^*}{v_n|\cos\overline{\theta}_2 - \cos\frac{3\overline{\theta}_1 + \overline{\theta}_2}{4}|}\rceil + T_1^* \leq \frac{C}{v_n}$, the distance between the leader agent i ($i \in V_2$) and the follower agent j ($j \in V_3$) satisfies the following relation:

$$\begin{aligned} d_{ij}(T_2^*+1) &\geq \Big|x_i(0) - x_j(0) + v_n\sum_{s=0}^{T_2^*}(\cos\overline{\theta}_2 - \cos\theta_j(s))\Big| \\ &\geq v_n\Big|\sum_{s=T_1^*+1}^{T_2^*}\cos\overline{\theta}_2 - \cos\theta_j(s))\Big| - v_n\Big|\sum_{s=0}^{T_1^*}(\cos\overline{\theta}_2 - \cos\theta_j(s))\Big| - |x_i(0) - x_j(0)| \\ &\geq v_n(T_2^* - T_1^*)\Big|\cos\overline{\theta}_2 - \cos\frac{3\overline{\theta}_1 + \overline{\theta}_2}{4}\Big| - 2v_nT_1^* - 1 \\ &\geq r_n. \end{aligned} \tag{4.8}$$

Since the leaders in subgroup $S2$ will not fall in the neighborhood of the followers, the heading of the followers will not be affected by the reference signal $\overline{\theta}_2$. By the heading update equation (2.8), the heading of the followers satisfies

$$\theta_j(T_2^*+1)\begin{cases} \geq \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} > \overline{\theta}_2 & \text{if } \overline{\theta}_2 < \overline{\theta}_1, \\ \leq \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} < \overline{\theta}_2 & \text{if } \overline{\theta}_2 > \overline{\theta}_1. \end{cases} \tag{4.9}$$

Repeating the above process, it can be concluded that (4.8) and (4.9) hold for all $t \geq T_2^*+1$.

Case II: $\pi < \overline{\theta}_2 < 2\pi$. For the follower $j \in V_3$, by Lemma 4.3, its heading at time $t \in [T_1^*+1, T_2^*]$ satisfies $0 \leq \theta_j(t) \leq \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} < \pi \leq \overline{\theta}_2$. Taking $T_3^* = \lceil \frac{1 + r_n + 2v_nT_1^*}{v_n|\sin\frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} - \sin\overline{\theta}_2|}\rceil + T_1^* \leq \frac{C}{v_n}$, for the leader agent i ($i \in V_2$) and the follower agent

j ($j \in V_3$), their distance satisfies the following inequality:

$$
\begin{aligned}
d_{ij}(T_3^*+1) &\geq \left| y_i(0) - y_j(0) + v_n \sum_{s=0}^{T_3^*} (\sin \overline{\theta}_2 - \sin \theta_j(s)) \right| \\
&\geq v_n \left| \sum_{s=T_1^*+1}^{T_3^*} (\sin \overline{\theta}_2 - \sin \theta_j(s)) \right| - 2v_n T_1^* - |y_i(0) - y_j(0)| \\
&\geq v_n (T_3^* - T_1^*) \left| \sin \frac{\overline{3\theta}_1 + \overline{\theta}_2}{4} - \sin \overline{\theta}_2 \right| - 2v_n T_1^* - 1 \\
&\geq r_n. \qquad (4.10)
\end{aligned}
$$

The followers will not have leader neighbors of subgroup $S2$ for $t = T_3^* + 1$, and the headings of followers satisfy

$$\theta_j(T_3^*+1) \leq \frac{3\overline{\theta}_1 + \overline{\theta}_2}{4} < \overline{\theta}_2. \qquad (4.11)$$

Repeating the above process, we conclude that (4.10) and (4.11) hold for all $t \geq T_3^* + 1$. This completes the proof of the lemma by taking $T^* = \max\{T_2^*, T_3^*\}$. □

Proof of Theorem 2.10. We first prove that the distance between any two agents $i \in V_1 \bigcup V_3$ and $j \in V_1 \bigcup V_3$ satisfies the inequality (4.2) for all $t \geq 0$. In Lemma 4.2, we have shown that (4.2) holds for $t \leq T^*$, and we assume that it holds for $T^* \leq t \leq t_0$.

By Lemma 4.4, we see that the followers have no leader neighbors in subgroup $S2$ for $t \geq T^*$. Thus, the headings of followers satisfy

$$
\begin{aligned}
\max_{i \in V_3} |\widetilde{\theta}_i(t+1)| &= \max_{i \in V_3} \left| \frac{\sum_{j \in \mathcal{N}_{i1}(t) \bigcup \mathcal{N}_{i3}(t)} \theta_j(t)}{n_{i1}(t) + n_{i3}(t)} - \overline{\theta}_1 \right| \\
&\leq \left(\max_{i \in V_3} \frac{n_{i3}(t)}{n_{i1}(t) + n_{i3}(t)} \right) \max_{i \in V_3} \widetilde{\theta}_i(t). \qquad (4.12)
\end{aligned}
$$

Using the induction assumption, for the follower i ($\in V_3$), the degrees of followers have the following bounds for $t \leq t_0$, $n_{i1}(t) + n_{i3}(t) \geq n_{i1}(0) + n_{i3}(0) - \max_{i \in V_3} c_i \geq nA_i(1 + o(\alpha_{1n})) - 4n\pi r_n^2 \frac{\alpha_{1n}}{64}(1 + o(1)) = nA_i(1 - \frac{\pi r_n^2 \alpha_{1n}}{16A_i})(1 + o(\alpha_{1n})) \geq nA_i(1 - \frac{\alpha_{1n}}{4}(1 + o(1)))$, and $n_{i3}(t) \leq n_{i3}(0) + \max_{i \in V_3} c_{i3} \leq n(1 - \alpha_{1n} - \alpha_{2n})A_i(1 + o(\alpha_{1n})) + 4n\pi \frac{\alpha_{1n}}{64} r_n^2 (1 + o(1)) \leq nA_i(1 - \frac{3}{4}\alpha_{1n}(1 + o(1)))$. Substituting these two inequalities into (4.12), we obtain

$$\max_{i \in V_3} |\widetilde{\theta}_i(t+1)| \leq \left(1 - \frac{\alpha_{1n}}{2}(1 + o(1))\right) \max_{i \in V_3} |\widetilde{\theta}_i(t)|.$$

Thus, the distance between any two agents $i \in V_1 \bigcup V_3$ and $j \in V_1 \bigcup V_3$ satisfies the following inequality:

$$\begin{aligned}
&\sum_{s=0}^{t_0} |d_{ij}(s+1) - d_{ij}(s)| \leq 2v_n + 2v_n \sum_{s=1}^{t_0} \max_{i \in V_3} |\widetilde{\theta}_i(s)| \leq 2v_n \\
&\quad + 2v_n \left\{ \sum_{s=1}^{t_0} (\widetilde{\alpha}_{1n})^{s-1} \max_{i \in V_3} |\widetilde{\theta}_i(1)| + \left[1 + (1 + \widetilde{\alpha}_{1n}) + \cdots + (1 + \widetilde{\alpha}_{1n} + \widetilde{\alpha}_{1n}^{T^*-2}) \right] \widetilde{\nu}_n \right\} \\
&\leq 2v_n \left(1 + \frac{2}{\alpha_{1n}} (|\overline{\theta}_1| + f_n) + \frac{C}{v_n \alpha_{1n}} \frac{4\alpha_{2n}}{\pi r_n^2} |\overline{\theta}_2 - \overline{\theta}_1| \right) \\
&\leq \frac{\alpha_{1n} r_n}{64}.
\end{aligned}$$

By mathematical induction, it is clear that (4.2) holds for all $t \geq 0$. By (4.12), the headings of followers have the following property:

$$\max_{i \in V_3} |\widetilde{\theta}_i(t+1)| \leq \left(1 - \frac{\alpha_{1n}}{2}(1 + o(1)) \right)^t \max_{i \in V_3} |\widetilde{\theta}_i(1)| \to 0, \text{as } t \to \infty.$$

This completes the proof of Theorem 2.10. ∎

5. Concluding remarks. In this paper, we consider the consensus decision-making problem of spatially distributed MASs where the neighbor relations are defined via the spatial relative positions. We obtained some quantitative results for the proportion of leaders needed in reaching the stable consensus for two cases: leaders with the same preference and leaders with different preferences. Our results can provide theoretical explanations for the observations in biological systems, and the analytical methods may be applied to analyze engineering systems and social networks. Some open problems deserve to be further investigated, for example, how to analyze a spatial distributed system when the proportions of two subgroups of leaders are equal? If the agents obey the three local rules for flocking, attraction, alignment, and repulsion, what is the proportion of leaders needed for a stable consensus?

Appendix A. Properties of the initial degrees and headings (I). We first introduce some results on the estimation of the initial leader degree and the follower degree for the system where the leaders have the same preference.

LEMMA A.1 (see [36]). *If the neighborhood radius r_n satisfies $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$, then the following assertions hold almost surely for large n:*

$$\begin{aligned}
&\max_{i \in V} \left| n_{i1}(0) - \sum_{j \in V_1} E\{ I(j \in N_{i1}(0)) \big| X_i(0) \} \right| = O(\phi_n), \\
&\max_{i \in V} \left| n_{i2}(0) - \sum_{j' \in V_2} E\{ I(j' \in N_{i2}(0)) \big| X_i(0) \} \right| = O(\varphi_n),
\end{aligned}$$

where $\phi_n = (n\alpha_n r_n^2 \log n)^{1/2}$, $\varphi_n = (nr_n^2 \log n)^{1/2}$, and $E\{\cdot|\cdot\}$ denotes the conditional expection.

COROLLARY A.2. *Suppose that the neighborhood radius satisfies the condition used in Lemma A.1. If the proportion of leaders satisfies $\frac{\log n}{nr_n^2} \ll \alpha_n \ll 1$, then* *the leader degree and follower degree of all agents at $t = 0$ satisfy*

(1) $\frac{n_{i1}(0)}{n_i(0)} = \alpha_n(1 + o(1))$, *a.s.*,

(2) $\frac{1}{4}n\alpha_n\pi r_n^2(1 + o(1)) \le \min_{i\in V} n_{i1}(0) \le \max_{i\in V} n_{i1}(0) \le n\alpha_n\pi r_n^2(1 + o(1))$, *a.s.*,

(3) $\frac{1}{4}n\pi r_n^2(1 + o(1)) \le \min_{i\in V} n_{i2}(0) \le \max_{i\in V} n_{i2}(0) \le n\pi r_n^2(1 + o(1))$, *a.s.*

Proof. By Lemma A.1, it is clear there exist two positive constants C and C' such that $\sum_{j\in V_1} E\{I(j \in N_{i1}(0))\big|X_i(0)\}-C\phi_n \le n_{i1}(0) \le \sum_{j\in V_1} E\{I(j \in N_{i1}(0))\big|X_i(0)\}+C\phi_n$ and $\sum_{j'\in V_2} E\{I(j' \in N_{i2}(0))\big|X_i(0)\} - C'\varphi_n \le n_{i2}(0) \le \sum_{j'\in V_2} E\{I(j' \in N_{i2}(0))\big|X_i(0)\} + C'\varphi_n$ hold almost surely for large n. By the independency and uniform distribution of the positions of all agents, we see that for the given initial position $X_i(0)$ of the agent i, $E\{I(j \in N_{i1}(0))\big|X_i(0)\}$ and $E\{I(j' \in N_{i2}(0))\big|X_i(0)\}$ are equal to the area of the neighborhood of agent i for any leader $j \in V_1$ and any follower $j' \in V_2$. Thus, $\frac{1}{4}\pi r_n^2 \le E\{I(j \in N_{i1}(0))\big|X_i(0)\} = E\{I(j' \in N_{i2}(0))\big|X_i(0)\} \le \pi r_n^2$. By this equation, we have $\frac{1}{4}n\alpha_n\pi r_n^2 \le \sum_{j\in V_1} E\{I(j \in N_{i1}(0))\big|X_i(0)\} \le n\alpha_n\pi r_n^2$ and $\frac{1}{4}n(1-\alpha_n)\pi r_n^2 \le \sum_{j'\in V_2} E\{I(j' \in N_{i2}(0))\big|X_i(0)\} \le n(1-\alpha_n)\pi r_n^2$. Using the conditions $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$ and $\frac{\log n}{nr_n^2} \ll \alpha_n \ll 1$, we have $\phi_n = o(n\alpha_n r_n^2 \log n)$ and $\varphi_n = o(nr_n^2 \log n)$. Hence, we obtain that

$$n_{i1}(0) = n\alpha_n E\{I(j \in N_{i1}(0))\big|X_i(0)\}(1 + o(1)),$$
$$n_{i2}(0) = n(1-\alpha_n)E\{I(j' \in N_{i2}(0))\big|X_i(0)\}(1 + o(1)).$$

Therefore, by these two equations and the above analysis, we have

$$\frac{n_{i1}(0)}{n_{i2}(0)} = \frac{n\alpha_n E\{I(j \in N_{i1}(0))\big|X_i(0)\}(1 + o(1))}{n(1-\alpha_n)E\{I(j' \in N_{i2}(0))\big|X_i(0)\}(1 + o(1))} = \alpha_n(1 + o(1)),$$

and the inequalities (2) and (3) can be easily obtained. ∎

Remark A.3. If the proportion of leaders satisfies $\alpha_n \ll \frac{\log n}{nr_n^2}$, then assertions (1) and (2) of the corollary do not hold, but the assertion (3) for the estimation of the followers still holds.

In the following, we will estimate the properties of headings at $t = 1$. According to the update equation of headings, for any $i \in V_2$, $\theta_i(1) = \sum_{j\in N_i(0)} \theta_j(0) = \sum_{j\in V} I_{j\in N_i(0)}\theta_j(0)$. It is clear that for different j_1 and j_2, the random variables $I_{j_1\in N_{i1}(0)}\theta_{j_1}(0)$ and $I_{j_2\in N_{i1}(0)}\theta_{j_2}(0)$ are not independent. Thus, the law of large numbers is not suitable to deal with such a case. By introducing a martingale difference sequence, we use the multiarray martingale estimation theorem to obtain the following theorem.

Lemma A.4. *If the neighborhood radius satisfies $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$, then the following assertions hold almost surely for large n:*

$$\max_{i\in V_1} |\theta_i(1) - \omega\bar{\theta}_0| \le (1 - \omega)L_n,$$
$$\max_{i\in V_2} |\theta_i(1)| \le L_n,$$

where $L_n = 4\sqrt{\frac{3\pi \log n}{nr_n^2}}(1 + o(1))$.

Proof. Set $\mathcal{F}_0(n) = \{\emptyset, \Omega\}$ and $\mathcal{F}_i(n) = \sigma\{\theta_j(0), (x_q(0), y_q(0)), j \le i, 1 \le q \le n\}$ with $1 \le i \le n$. Under Assumption 2.1, for $1 \le j \le n$, $I(j \in N_i(0))$ is $\mathcal{F}_i(n)$-measurable for all i. Moreover, for any $1 \le j \le n$, we have $E\{\theta_j(0)|\mathcal{F}_{j-1}(n)\} = E\theta_j(0) = 0$, and $E\{\theta_j^2(0)|\mathcal{F}_{j-1}(n)\} = E\theta_j^2(0) = \frac{\pi^2}{3}$. Thus, $\{\theta_j(0), \mathcal{F}_j(n), 1 \le j \le n\}$ is a martingale difference sequence with constant conditional variance.

Using Lemma A.1, we have

$$\max_{i\in V} \sum_{j=1}^{n} I\left(j \in N_i(0)\right) \le \max_{i\in V} n_{i1}(0) + \max_{i\in V} n_{i2}(0) \le n\pi r_n^2(1 + o(1)).$$

Set $c_n = \frac{2}{\pi}\sqrt{\frac{3\log n}{n\pi r_n^2}}$, using the multiarray martingale lemma (Lemma 7 in [15]), we have for large n

$$\begin{aligned}\max_{i\in V}\Big|\sum_{j\in N_i(0)} \theta_j(0)\Big| &= \frac{1}{c_n}\max_{i\in V}\left|\sum_{j=1}^{n} c_n I\left(j\in N_i(0)\right)\theta_j(0)\right| \\ &\le \frac{\pi^2}{4} n\pi r_n^2 c_n(1+o(1)) + \frac{3\log n}{c_n} = \pi\sqrt{3n\pi r_n^2\log n}(1+o(1)), \quad \text{a.s.}\end{aligned} \tag{A.1}$$

Therefore, by (2.2) and Corollary A.2, we have

$$\begin{aligned}\max_{i\in V_1}|\theta_i(1) - \omega\bar{\theta}_0| &= (1-\omega)\max_{i\in V_1}\left|\frac{\sum_{j\in N_i(0)}\theta_j(0)}{n_i(0)}\right| \\ &\le (1-\omega)\frac{\pi\sqrt{3n\pi r_n^2\log n}(1+o(1))}{\frac{1}{4}n\pi r_n^2(1+o(1))} = (1-\omega)L_n.\end{aligned}$$

Similarly, by (2.3), the headings of followers satisfy

$$\max_{i\in V_2}|\theta_i(1)| = \max_{i\in V_2}\left|\frac{\sum_{j\in N_i(0)}\theta_j(0)}{n_i(0)}\right| \le 4\sqrt{\frac{3\pi\log n}{nr_n^2}}(1+o(1)), \quad \text{a.s.}$$

∎

Appendix B. Properties of the initial degrees and headings (II). In this section, we estimate the initial leader degree, the follower degree, and the heading properties for the system where the leaders have different preferences.

LEMMA B.1. *Assume that the neighborhood radius r_n satisfies $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$, and the proportion of leaders in subgroup $S1$ satisfies $\alpha_{1n} \gg \sqrt{\frac{\log n}{nr_n^2}}$. Then the following assertions hold almost surely for large n:*

$$\begin{aligned}\max_{i\in V_3}\Big|n_i(0) - \sum_{j\in V} E\{I(j\in\mathcal{N}_i(0))\big|X_i(0)\}\Big| &\le C_1\varphi_n,\\ \max_{i\in V_3}\Big|n_{i1}(0) - \sum_{j\in V_1} E\{I(j\in\mathcal{N}_{i1}(0))\big|X_i(0)\}\Big| &\le C_2\psi_n,\\ \max_{i\in V_3}\Big|n_{i3}(0) - \sum_{j\in V_3} E\{I(j\in\mathcal{N}_{i3}(0))\big|X_i(0)\}\Big| &\le C_3\varphi_n,\end{aligned}$$

where C_1, C_2, and C_3 are positive constants independent of n, $\varphi_n = (nr_n^2\log n)^{1/2}$, and $\psi_n = (n\alpha_{1n}r_n^2\log n)^{1/2}$, and $E\{\cdot|\cdot\}$ denotes the conditional expectation.

The above lemma can be obtained by following the proof of Lemma 11 in [36], and we omit the proof details.

Remark B.2. Denote A_i as the area of the circle lying in the unit square centered at $X_i(0)$ with the radius r_n. For any agent j, we have $\frac{\pi r_n^2}{4} \le E\{I(j \in N_i(0))|X_i(0)\} = E\{I(j \in N_{i1}(0))\big|X_i(0)\} = E\{I(j \in N_{i3}(0))\big|X_i(0)\} = A_i \le \pi r_n^2$.

LEMMA B.3. *If the neighborhood radius satisfies* $\sqrt{\frac{\log n}{n}} \ll r_n \ll 1$, *and the proportion of leaders in subgroups* $S1$ *and* $S2$ *satisfies* $\alpha_{1n} \gg \sqrt{\frac{\log n}{nr_n^2}}$ *and* $\alpha_{2n} \ll v_n r_n \sqrt{\frac{\log n}{n}}$, *respectively, then we have for large* n

$$\max_{i\in V_3} |\widetilde{\theta}_i(1)| \le (|\overline{\theta}_1| + f_n)(1+o(1)), \quad a.s.,$$

where $f_n = O(\sqrt{\frac{\log n}{nr_n^2}})$.

Proof. Using methods similar to that of (A.1), the term of the averaged heading can be estimated by $\max_{i\in V_3} |\sum_{j\in N_{i3}(0)} \theta_j(0)| \le \pi\sqrt{3n\pi r_n^2 \log n}(1+o(1))$ almost surely. By (2.8) we have, for the follower i,

$$\max_{i\in V_3} |\widetilde{\theta}_i(1)| \le \max_{i\in V_3} \frac{n_{i2}(0)|\overline{\theta}_2 - \overline{\theta}_1| + |\sum_{j\in \mathcal{N}_{i3}(0)} \theta_j(0)| + n_{i3}(0)|\overline{\theta}_1|}{n_i(0)}.$$

Case I: $\overline{\theta}_1 \neq 0$. By the conditions on r_n, α_{1n}, and α_{2n}, we have $n_{i3}(0) \ge \frac{n(1-\alpha_{1n}-\alpha_{2n})\pi r_n^2(1+o(1))}{4} \gg \sqrt{n\pi r_n^2 \log n} \gg v_n r_n \sqrt{n \log n} \gg n\alpha_{2n} \ge n_{i2}(0)$. Thus, we have

$$\begin{aligned}\max_{i\in V_3} |\widetilde{\theta}_i(1)| &= \max_{i\in V_3} \frac{n_{i3}(0)|\overline{\theta}_1|}{n_i(0)}(1+o(1)) \\ &= \max_{i\in V_3} \frac{n(1-\alpha_{1n}-\alpha_{2n})A_i}{nA_i}|\overline{\theta}_1|(1+o(1)) \le |\overline{\theta}_1|(1+o(1)).\end{aligned}$$

Case II: $\overline{\theta}_1 = 0$. By the conditions of the lemma, we have $n_{i2}(0) \le n\alpha_{2n} \ll \sqrt{nr_n^2 \log n}$. Thus,

$$\begin{aligned}\max_{i\in V_3} |\widetilde{\theta}_i(1)| &\le \max_{i\in V_3} \frac{\pi\sqrt{3n\pi r_n^2 \log n}(1+o(1)))}{nA_i} \\ &\le \frac{\pi\sqrt{3n\pi r_n^2 \log n}(1+o(1))}{\frac{n\pi r_n^2(1+o(1))}{4}} \\ &= 4\pi\sqrt{\frac{3\log n}{n\pi r_n^2}}(1+o(1)).\end{aligned}$$ ∎

Following the proof of Lemma 11 in [36], we can estimate the number of agents in the sets $\mathcal{C}_i$ and $\mathcal{C}_{i3}$.

LEMMA B.4. *Under the conditions of Lemma* B.3 *we have, for large* n,

$$\max_{i\in V_3} \Big| c_i - \sum_{j\in V} E\{I(j\in \mathcal{C}_i)|X_i(0)\}\Big| \le C_4\sqrt{nr_n^2 \log n}, \quad a.s.,$$

$$\max_{i\in V_3} \Big| c_{i3} - \sum_{j\in V_3} E\{I(j\in \mathcal{C}_{i3})|X_i(0)\}\Big| \le C_5\sqrt{nr_n^2 \log n}, \quad a.s.,$$

where C_4 *and* C_5 *are positive constants independent of* n.

REFERENCES

[1] C. Reynolds, *Flocks, herds, and schools: A distributed behavioral model*, Comput. Graph., 21 (1987), pp. 25–34.
[2] R. Axelrod, *The Evolution of Cooperation*, Basic Books, New York, 1984.
[3] F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, *Graph topology plays a determinant role in the evolution of cooperation*, Proc. B, 273 (2006), pp. 51–55.
[4] D. Helbing, I. Farkas, and T. Vicsek, *Simulating dynamic features of the escape panic*, Nature, 407 (2000), pp. 487–490.
[5] A. Czirok, E. B. Jacob, I. Cohen, O. Schochet, and T. Vicsek, *Formation of complex bacterial colonies via self-generated vortices*, Phys. Rev. E(3), 54 (1996), pp. 1791–1801.
[6] A. Mirtabatabaei and F. Bullo, *Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems*, SIAM J. Control Optim., 50 (2012), pp. 2763–2785.
[7] C. Altafini and G. Lini, *Predictable dynamics of opinion forming for networks with antagonistic interactions*, IEEE Trans. Automat. Control, 60 (2015), pp. 342–357.
[8] R. Olfati-Saber, *Flocking for multi-agent dynamic systems: Algorithms and theory*, IEEE Trans. Automat. Control, 51 (2006), pp. 401–420.
[9] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett., 75 (1995), pp. 1226–1229.
[10] M. Cao, C. Yu, and B. D. O. Anderson, *Formation control using range-only measurements*, Automatica J. IFAC, 47 (2011), pp. 776–781.
[11] F. Döfler, M. Chertkovb, and F. Bullo, *Synchronization in complex oscillator networks and smart grids*, Proc. Natl. Acad. Sci., 110 (2013), pp. 2005–2010.
[12] S. Liu, T. Li, and L. Xie, *Distributed consensus for multiagent systems with communication delays and limited data rate*, SIAM J. Control Optim., 49 (2011), pp. 2239–2262.
[13] Q. Zhang and J.-F. Zhang, *Quantized data–based distribued consensus under directed time-varying communication topology*, SIAM J. Control Optim., 51 (2013), pp. 332–352.
[14] A. Jadbabaie, J. Lin, and A. S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Automat. Control, 48 (2003), pp. 988–1001.
[15] Z. X. Liu and L. Guo, *Synchronization of multi-agent systems without connectivity assumption*, Automatica J. IFAC, 45 (2009), pp. 2744–2753.
[16] G. Chen, Z. Liu, and L. Guo, *The smallest possible interaction radius for synchronizaiton of self-propelled particles*, SIAM Rev., 56 (2014), pp. 499–521.
[17] S. G. Reebs, *Can a minority of informed leaders determine the foraging movements of a fish shoal?*, Animal Behav., 59 (2000), pp. 403–409.
[18] A. Stroeymeyt, N. R. Franks, and M. Giurfa, *Knowledgeable individuals lead collective decisions in ants*, J. Exp. Biol., 214 (2011), pp. 3046–3054.
[19] I. D. Couzin, J. Krause, N. R. Franks, and S. Levy, *Effective leadership and descion-making in animal groups on the move*, Nature, 433 (2005), pp. 513–516.
[20] Y. Cao, W. Ren, and Y. Li, *Distributed discrete-time coordinated tracking with a time-varying reference state and limited communication*, Automatica J. IFAC, 45 (2009), pp. 1299–1305.
[21] Z. K. Li, X. D. Liu, W. Ren, and L. H. Xie, *Distributed tracking control for linear multiagent systems with a leader of bounded unknown input*, IEEE Trans. Automat. Control, 58 (2013), pp. 518–523.
[22] W. W. Yu, G. R. Chen, and M. Cao, *Distributed leader-follower flocking control for multi-agent dynamical systems with time-varying velocities*, Systems Control Lett., 59 (2010), pp. 543–552.
[23] Z. Y. Meng, D. V. Dimarogonas, and K. H. Johansson, *Leader-follower coordinated tracking of multiple heterogeneous Lagrange systems using continuous control*, IEEE Trans. Robo., 30 (2014), pp. 739–745.
[24] P. Gupta and P. R. Kumar, *Critical power for asymptotic connectivity in wireless networks*, in Stochastic Analysis, Control, Optimization and Applications, Birkhauser, Boston, pp. 547–566.
[25] F. Xue and P. R. Kumar, *The number of neighbors needed for connectivity of wireless networks*, Wirel. Netw., 10 (2004), pp. 169–181.
[26] J. Cortés, S. Martínez, and F. Bullo, *Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions*, IEEE Trans. Automat. Control, 51 (2006), pp. 1289–1298.
[27] G. Notarstefano, K. Savla, F. Bullo, and A. Jadbabaie, *Maintaining limited-range connectivity among second-order agents*, Proceedings of the 2006 American Control Conference, Minneapolis, MN, IEEE, Piscataway, NJ, 2006, pp. 2124–2129.

[28] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, *Distributed memoryless point convergence algorithm for mobile robots with limited visibility*, IEEE Trans. Robot. Autom., 15 (1999), pp. 818–828.

[29] L. Wang, X. Wang, and X. Hu, *Connectivity maintenance and distributed tracking for double-integrator agents with bounded potential functions*, Internat. J. Robust Nonlinear Control, 25 (2015), pp. 542–558.

[30] M. M. Zavlanos and G. J. Pappas, *Potential fields for maintaining connectivity of mobile networks*, IEEE Trans. Robot., 23 (2007), pp. 812–816.

[31] D. V. Dimarogonas and K. J. Kyriakopoulos, *Connectedness preserving distributed swarm aggregation for multiple kinematic robots*, IEEE Trans. Robot., 24 (2008), pp. —1223.

[32] T. Gustavi, D. V. Dimarogonas, M. Egerstedt, and X. Hu, *Sufficient conditions for connectivity maintenance and rendezvous in leader-follower networks*, Automatica J. IFAC, 46 (2010), pp. 133–139.

[33] C. Altafini and G. Lini, *Predictable dynamics of opinion forming for networks with antagonistic interactions*, IEEE Trans. Automat. Control, 60 (2015), pp. 342–357.

[34] B. Nabet, N. E. Leonard, I. D. Couzin, and S. A. Levin, *Dynamics of decision-making in animal group motion*, J. Nonlinear Sci., 19 (2009), pp. 399–435.

[35] N. E. Leonard, T. Shen, B. Nabet, L. Scardovi, I. D. Couzin, and S. A. Levin, *Decision versus compromise for animal groups in motion*, Proc. Natl. Acad. Sci. USA, 109 (2012), pp. 227–232.

[36] Z. X. Liu, J. Han, and X. M. Hu, *The proportion of leaders needed for the expected consensus*, Automatica J. IFAC, 47 (2011), pp. 2697–2703.

[37] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, Springer, New York, 1997.

[38] M. Penrose, *Random Geometric Graphs*, Oxford University Press, Oxford, 2003.

[39] B. Bollobás, *Random Graphs*, 2nd ed., Cambridge University Press, Cambridge, 2001.

[40] Z. X. Liu, *Consensus decision-making of animal groups in motion*, Proc. of World Congress on Intelligent Control and Automation, Beijing, China, IEEE, Piscataway, NJ, 2012, pp. 952–957.