A Comparison of System Optimum and User Equilibrium Traffic Assignment Using the Concave Form of the Fundamental Diagram

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Abstract— **This study deals with the implementation of the two types of Wardrop Equilibrium for traffic networks using the concave form of the Fundamental Diagram (FD). This incorporation of the FD is especially important for modeling congested traffic since previous work have used the piecewise linear form of FD which does not replicate the spatiotemporal traffic patterns. For this purpose, traffic is modeled after the cell-transmission model (CTM). A comprehensive comparison of the impacts of each type of equilibrium is provided for a sample network and applications of each type are explained.**

I. INTRODUCTION

There are two types of equilibrium points that can be computed: 1) *User Equilibrium* (UE): It states that the journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route. 2) *System Optimal Equilibrium* (SO): It states that at equilibrium, the average journey time is at a minimum. That implies that all users behave cooperatively in choosing their routes to ensure the most efficient use of the whole system [1]. One way to estimate the potential benefits of a *Dynamic Traffic Assignment* (DTA) model is to compare the total cost incurred. In fact, this approach has been widely used in static analyses of the traffic impacts of various DTA techniques. In this research, UE and SO methods for trip assignment are implemented. Then, link flows and Total Travel Times (TTT) under both conditions are compared. Having the travel times, and using Dijkstras algorithm, the shortest paths between all O-D pairs of the sample network are found. At the end, a through comparison of US and SO performance is provided.

II. METHODOLOGY

A. Cell Transmission Model (CTM)

This study chooses CTM as the underlying dynamic traffic flow model. The CTM model can be stated by the following two conditions [2]:

$$\frac{\partial f}{\partial x} + \frac{\partial k}{\partial t} = 0 \qquad and \qquad f = F(k, x, t) \tag{1}$$

where f is the traffic flow (veh/mi); k is the density (veh/h); x and t, respectively, are the position and time variables; and F is a function relating f and k. Cells are smaller sections of each link. The assumption here is that the initial density of each cell is known. These densities are chosen to have high values to bring the network close to congested situation

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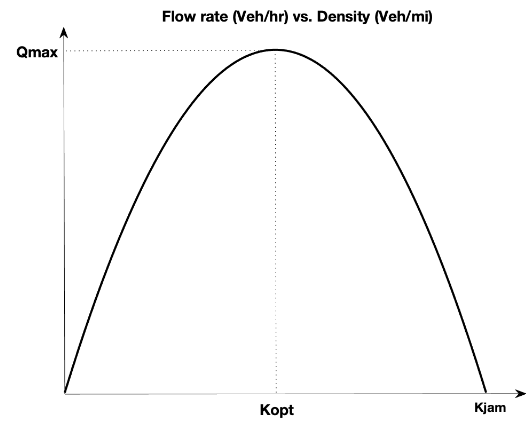

Fig. 1. Concave Estimate of Fundamental Diagram

and make the problem more challenging. The flow rate f in the CTM is a function of the density k. Therefore, by using the FD of each cell, the flow rate, f will also be found. So far, most traffic-related studies have used a piecewise linear estimate of this diagram. However, in this study, a concave estimate of it will be used (fig. 1) which makes the computations capable of reflecting the spatiotemporal features of the traffic patterns more realistically.

B. Capacity Restraint Method

The Bureau of Public Roads (BPR) developed a capacity restraint (also called link-congestion, volume-delay, or link performance) function which says for each link there is a function stating the relationship between resistance and volume of traffic [3]:

$$t_a(x_a) = t_{opt}(1 + 0.15(\frac{x_a}{c_a})^4) \tag{2}$$

where t_{opt} is free flow travel time on link a per unit of time, x_a is flow attempting to use link a, c_a is capacity of link a per unit of time and $t_a(x_a)$ is the average travel time for a vehicle on link a. Here, this function is used to calculate $t_a(x_a)$ of each link based on its flow x_a.

C. Wardrop Equilibrium Optimization Problems

The two Wardrop equilibrium optimization formulation is stated in this section [4]. Here, $\delta^{rs}_{a,k}$ is a definitional constraint given by Eq. (3). The SO optimization problem is formulated by Eq. (4).

$$\delta^{rs}_{a,k} = \begin{cases} 1 & \text{if link } a \text{ belongs to path } k \\ 0 & \text{Otherwise.} \end{cases} \tag{3}$$

$$\begin{aligned}
&\min\ S(x) = \sum_a x_a t_a(x_a) \\
&\text{subject to} \sum_k f_k^{rs} = q_{rs} \quad : \forall r, s \\
&x_a = \sum_r \sum_s \sum_k \delta_{a,k}^{rs} f_k^{rs} \quad : \forall a \\
&f_k^{rs} \geq 0 \quad : \forall k, r, s \\
&x_a \geq 0 \quad : a \in A
\end{aligned} \tag{4}$$

where k is path number, x_a is equilibrium flow in link a, t_a is travel time on link a, f_k^{rs} is flow on path k connecting O-D pair $r-s$, q_{rs} is trip rate between $r-s$. The UE optimization problems is formulated by Eq. (5):

$$\begin{aligned}
&\min\ S(x) = \sum_a \int_0^{x_a} t_a(x_a) \\
&\text{subject to} \sum_k f_k^{rs} = q_{rs} \quad : \forall r, s \\
&x_a = \sum_r \sum_s \sum_k \delta_{a,k}^{rs} f_k^{rs} \quad : \forall a \\
&f_k^{rs} \geq 0 \quad : \forall k, r, s \\
&x_a \geq 0 \quad : a \in A
\end{aligned} \tag{5}$$

III. Numerical Example and Results

To investigate the performance of SO and UE numerically, a sample network with 9 nodes and 13 links was chosen. Links are formed by different number of cells. Nodes 1, 2, 3 and 4 were origin nodes and node 5 was the destination node. MATLAB and *fmincon* function as the nonlinear programming solver was used to solve the SO and UE optimization problems. Via these methods, flow rates of the network cells were found. Having the flow rates, velocity of each cell was calculated by $V_{ij} = f_{ij}/k_{ij}$. Knowing the length of each cell, TTT was computed by $TTT_{ij} = L_{ij}/V_{ij}$. Having the TTT of all cells, TTT of links and paths were also calculated. Fig. 2 compares the TTT of SO and UE with optimum TTT of each link. Optimum TTT (TTT_{opt}) was found via the FD of each cell by finding the free flow speed $V_{max} = Q_{max}/K_{opt}$ and then using $TTT_{opt} = L_{ij}/V_{max}$. Since the initial densities of the cells are chosen to be high, as expected, it can be seen that TTT of both methods are much higher than the free flow (optimum) TTT of the network. Then, again in MATLAB, Dijkstra's algorithm was applied and among all the possible paths from each source to the destination, shortest paths and their cost (TTT) were found. TTT is in hour unit. Table I shows the results of this algorithm. Also, the overall performance of SO and UE method were compared. Results are shown in table II. Comparison showed that both methods were able to serve the same number of demands in the network, however, in SO method, the TTT of the network was much less than that of the UE method as expected. But, in UE method, since routing is performed to selfishly minimize individuals' travel time, it is seen that the sum of the shortest path costs is lower in comparison to that of the SO method.

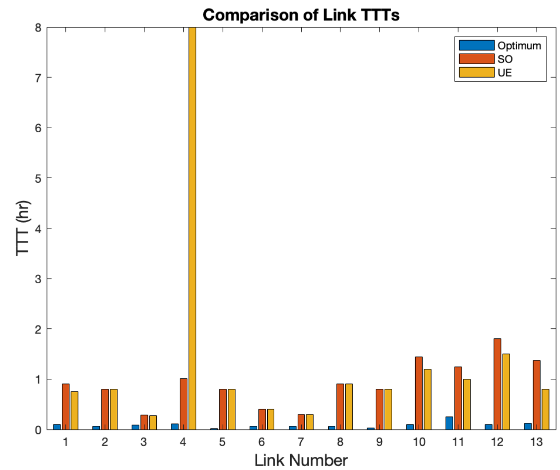

Fig. 2. Comparison of the optimum and calculated TTTs via SO and UE

TABLE I
TTT of Shortest Paths

O-D Pairs	UE	SO
O 1- D 5	1.4700	1.4149
O 2- D 5	1.9000	2.2054
O 3- D 5	2.4000	2.6443
O 4- D 5	1.9000	2.1428

TABLE II
Performance Comparison of SO and UE

O/D Pair	Optimum	UE	SO
Total Served Demand (veh/hr)	910.0000	910.0000	910.0000
Network TTT (hr)	1.1761	17.5200	12.0675
Total Shortest Path TTT (hr)	1.1760	7.6700	8.4074

IV. CONCLUSIONS

In this study, the concave form of FD was used which represents the spatiotemporal features of the traffic patterns more realistically comparing with its piecewise linear form. The SO and UE methods were implemented and compared numerically for a sample network. Comparing the total cost of the two methods as a measure of the magnitude of the inefficiency that is brought about by congestion represents the possible benefits of each method that can be realized and how their different performance influences both the network and individual drivers behavior.

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