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A STUDY ON DISCRETE MULTISCALE EDGE REPRESENTATIONS

Zeev Berman and John S. Baras

Electrical Engineering Department and Systems Research Center
University of Maryland, College Park, MD 20742 , USA
e-mail: berman@src.umd.edu

Abstract[1]

The analysis of a discrete multiscale edge representation is considered. A general signal description, called an inherently bounded Adaptive Quasi Linear Representation (AQLR), motivated by two important examples, namely, the wavelet maxima representation, and the wavelet zero-crossings representation, is introduced. This paper addresses the questions of uniqueness, stability, and reconstruction. It is shown, that the dyadic wavelet maxima (zero-crossings) representation is, in general, nonunique. Nevertheless, these representations are always stable. Using the idea of the inherently bounded AQLR, two stability results are proven. For a general perturbation, a global BIBO stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied. A reconstruction algorithm, based on the minimization of an appropriate cost function, is proposed.

1 Introduction

An interesting and promising approach to a signal representation is to make explicit important features in the data. The first example, taught in elementary calculus, is a "sketch" of a function based on extrema of a signal and possibly of its first few derivatives. The second instance, widely used in computer vision, is an edge representation of an image. If the size of an expected feature is apriori unknown, a need for a multiscale analysis is apparent. Therefore, it is not surprising that multiscale sharper variation points (edges) are meaningful features for many signals, and they have been applied, for example, in edge detection, signal compression, pattern classification, pattern matching, and speech analysis.

Traditionally, multiscale edges are determined either as extrema of Gaussian-filtered signals [9] or as zero-crossings of signals convolved with the Laplacian of a Gaussian (see e.g. [4] for a comprehensive review). S. Mallat in series of papers [8, 6, 7] introduced zero-crossings and extrema of the wavelet transform as a multiscale edge representation. Two important advantages of these methods are low algorithmic complexity and flexibility in choosing the basic filter. Moreover, [6] and [7] propose reconstruction procedures and show accurate numerical reconstruction results from zero-crossings and maxima representations. In [6, 7], as in many other works in this area, the basic algorithms were developed using continuous variables. The continuous approach gives an excellent background to motivate and justify the use of either local extrema or zero-crossings as important signal features. Unfortunately, in the continuous framework, analytic tools to investigate the information content of the representation are not yet available. The knowledge about properties of the representations is mainly based on empirical reconstruction results. From the theoretical point of view, there are still important open problems, e.g. stability, uniqueness, and structure of a reconstruction set (a family of signals having the same representation).

Our objective is to analyze these theoretical questions using a model of an actual implementation. The main assumption is that the data is discrete and finite. The discrete multiscale maxima and zero-crossings representations are defined in a general set-up of a linear filter bank, however, the main goal is to consider a particular case where the filter bank describes the wavelet transform. Since reconstruction sets of both maxima and zero-crossings representations have a similar structure, a general form called the Adaptive Quasi Linear Representation (AQLR) is introduced. Moreover, many generalizations of the basic maxima and zero-crossings representations fit into the framework of the AQLR. This paper uses the idea of the AQLR to investigate rigorously three fundamental questions: uniqueness, stability, and reconstruction.

We first present conditions for uniqueness, then apply these conditions to the wavelet transform-based representation, and obtain a conclusive result. It turns out, that neither the wavelet maxima representation nor the wavelet zero-crossings representation is , in general, unique. The proof is based on constructing a sinusoidal sequence, whose maxima (zero-crossings) representation cannot be unique for any dyadic wavelet transform.

The next subject is stability of the representation. This issue is of great importance because there are many known examples of unstable zero-crossings representations. In order to improve the stability properties, Mallat has included additional sums in the standard zero-crossings representation and, together with Zhong, they have introduced the wavelet maxima representation, as a stable alternative to

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the zero-crossings representation. Indeed, very good numerical results have been reported, but stability analysis has not been pursued. Using the idea of the inherently bounded AQLR, we are able to prove stability results. For a general perturbation, global BIBO (bounded input, bounded output) stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied.

One of the most important practical problems is the need for an effective reconstruction scheme. Mallat and Zhong [7] and Mallat [6] have used algorithms based on alternate projections. In this paper, an alternative reconstruction scheme is proposed. The procedure is valid for any inherently bounded AQLR and it is based on an appropriate cost function, whose minimum is achieved at the reconstruction set. Specifically, we focus on an algorithm which is based on the integration of the gradient of the cost function. It is shown that this algorithm approaches the reconstruction set. This method yields efficient, parallel algorithms, which are especially promising in the case of the wavelet-based representation. In particular, an analog-hardware implementation, similar to a neural network, may lead to a very efficient and fast scheme.

2 Multiscale Maxima Representation

This section presents the definitions of a discrete multiscale extrema (maxima) representation, an Adaptive Quasi Linear Representation (AQLR), and an inherently bounded AQLR. The main result is to show that the multiscale maxima representation, based on a wavelet transform, is an inherently bounded AQLR.

Consider $\mathcal{L}$, a linear space of real, finite sequences:

$$\mathcal{L} \triangleq \left\{ f : f = \{f(n)\}_{n=0}^{N-1} \;\; f(n) \in \Re \right\}.$$

Let X and Y denote operators on $\mathcal{L}$ which provide the sets of local maxima and minima, respectively, of a sequence $f \in \mathcal{L}$. The formal definitions are :

$$Xf = \{k : f(k+1) \leq f(k) \text{ and } f(k-1) \leq f(k)\}$$

$$Yf = \{k : f(k+1) \geq f(k) \text{ and } f(k-1) \geq f(k)\}$$

In this work, in order to avoid boundary problems, an N-periodic extension of finite sequences is assumed.

Let $W_1, W_2, \ldots, W_J, S_J$ be linear operators on $\mathcal{L}$. The sets XW_jf, YW_jf are local maxima and minima points of the sequence W_jf. The values of W_jf at extreme points are denoted by $\{W_jf(k)\}_{k \in EW_jf}$ where $EW_jf = XW_jf \cup YW_jf$. The multiscale local extrema representation, R_mf is defined as:

$$R_mf \triangleq \left\{ \left\{ XW_jf, YW_jf, \{W_jf(k)\}_{k \in EW_jf} \right\}_{j=1}^{J}, S_Jf \right\}.$$

Following [7], R_mf, will be called the multiscale maxima representation as well. In the particular case, when $W_1, W_2, \ldots, W_J, S_J$ correspond to a wavelet transform, R_mf will be called the wavelet maxima representation.

The determination of the extrema point sets is highly nonlinear. However, for the given extrema sets, XW_jf and YW_jf, the remaining data are obtained by a linear operation of sampling an image of a linear operator at fixed points. This observation is the motivation to consider R_mf as consisting of two parts: the sampling information and the maxima information. The sampling information is the sequence S_Jf and the values of W_jf at the points $XW_jf \cup YW_jf$ (j=1,2,...,J). The maxima information consists of the sets XW_jf, YW_jf and of the fact that the elements of XW_jf and YW_jf are local maxima and minima of W_jf.

Let T_{mf} denote the linear operator associated with the sampling information. Then, R_mf is written in an alternative way as:

$$R_mf = \left\{ \{XW_jf, YW_jf\}_{j=1}^{J}, T_{mf}f \right\}. \tag{1}$$

For a given representation Rf, a reconstruction set $\Gamma(Rf)$ is defined as a set of all sequences satisfying this representation, i.e.

$$\Gamma(Rf) \triangleq \{\gamma \in \mathcal{L} : R\gamma = Rf\}. \tag{2}$$

At this point, the structure of the reconstruction set of the multiscale maxima representation is considered. It is clear that in order to satisfy a given maxima representation, a sequence $h \in \mathcal{L}$, in addition to obeying the sampling information $T_{mf}h = T_{mf}f$, needs to meet the requirement that W_jh has local extrema at the points of XW_jf and YW_jf. Loosely speaking, we have to assure that W_jh is increasing after a minimum and before a maximum and it is decreasing otherwise. Since rigorous description requires several definitions which will not be used in the sequel in this paper, let us omit these details and present the result.

Theorem 1 *R_mf is a given multiscale maxima representation. $h \in \Gamma(R_mf)$ if and only if*

$$T_{mf}h = T_{mf}f \tag{3}$$

$$t(k) \cdot (W_jh(k+1) - W_jh(k)) > 0. \tag{4}$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for almost all k (if two consecutive k's belong to EW_jf then the first is omitted here). $t(k)$ is called the type of k and can be either $+1$ or -1.

The maxima representations can be cast into the form $Rf = \{Vf, Tf\}$, where Vf is a set of points and T is a linear operator which may depend on Vf. However, the key feature of the maxima representation is the fact that the set Vf yields additional constraints, in the form of linear inequalities, which do not appear directly in Rf. Stimulated by this observation, we define the following general family of signal representations.

Definition 1 *$Rf = \{Vf, Tf\}$ is called an Adaptive Quasi Linear Representation (AQLR) if there exists a linear operator A and a sequence a such that:*

$$x \in \Gamma(Rf) \Leftrightarrow Tx = Tf \text{ and } Ax > a. \tag{5}$$

A, a may depend on Vf, but they must be independent of Tf.

The reasoning behind the name "Adaptive Quasi Linear Representation" (AQLR) is as follows. This representation is adaptive since T, A, a depend on the sequence f (via the set Vf). It is quasi linear because it is based on a set of linear equalities and inequalities.

Clearly, the following is true.

Proposition 1 *Any multiscale maxima representation is an AQLR.*

The next definition is a generalization of an essential boundness property of the wavelet maxima representation.

Definition 2 *An AQLR is called inherently bounded if there exists a real $K > 0$ such that*

$$x \in \Gamma(Rf) \Rightarrow \|x\| \leq K\|Tf\|.$$

In this work, $\|\cdot\|$ denotes the Euclidean norm. The coefficient K can depend on the parameters of the representation e.g. $N, J, W_1, \ldots, W_J, S_J$ but it must be independent of Vf and Tf.

Proposition 2 *The wavelet maxima representation is an inherently bounded AQLR.*

Proof: Let $h \in \Gamma(R_m f)$. We need to find $K > 0$ such that $\|h\| \leq K\|T_{mf} f\|$. First recall (or see [7]) that the discrete wavelet transform satisfies Parseval's equality, namely:

$$\|h\|^2 = \|S_J h\|^2 + \sum_{j=1}^{J} \|W_j h\|^2. \tag{6}$$

Therefore, it suffices to bound $\|W_j h\|$, $\|S_J h\|$. $S_J h$ is included in $T_{mf} f$, hence:

$$\|S_J h\| \leq \|T_{mf} f\|. \tag{7}$$

Consider:

$$|\, W_j h(n) \,| \leq \max_n |\, W_j h(n) \,| = \max_n |\, W_j f(n) \,| \leq \|T_{mf} f\|.$$

The middle equality holds because $W_j h$ has the same local extrema as $W_j f$, in particular it has the same global extrema as $W_j f$. The right inequality is valid since $\max_n |\, W_j h(n) \,|$ appears (with its original sign) as a component of $T_{mf} f$. Therefore we conclude

$$\|W_j h\| \leq \sqrt{N}\|T_{mf} f\|. \tag{8}$$

Substituting (8) and (7) to (6) yields:

$$\|h\| \leq \sqrt{(NJ+1)}\|T_{mf} f\|. \tag{9}$$

□

Remark: The above bound is not the best possible, for example the factor $\sqrt{J}$ can easily be removed. However, we conjecture that the best bound has to be of the order of $\sqrt{N}\|T_{mf} f\|$.

The above theorem is key for our work. The subsequent results concerning stability and reconstruction will be developed in the framework of the inherently bounded AQLR. On the other hand, one can easily learn from the proof how to generalize the basic maxima representation, while maintaining the property of the inherently bounded AQLR.

3 Multiscale Zero-Crossings

In defining the multiscale zero-crossings representation, we essentially follow [6], but minor changes are necessary due to our basic assumption that only a discrete signal version is available. Let Z be an operator which provides the set of zero-crossings of a given sequence $f \in \mathcal{L}$, i.e.

$$Zf \triangleq \{k : f(k-1) \cdot f(k) \leq 0\}. \tag{10}$$

Mallat in [6] has stabilized the zero-crossings representation by including the values of the wavelet transform integral calculated between consecutive zero-crossing points. Therefore, the multiscale zero-crossings representation, $R_z f$, is defined as:

$$R_z f \triangleq \left\{\{ZW_j f, U_j^{zf} f\}_{j=1}^{J}, S_J f\right\}. \tag{11}$$

where

$$U_j^{zf} f(k) = \sum_{l=k}^{n(k)-1} W_j f(l).$$

k and $n(k)$ are two consecutive zero-crossings of $W_j f$.

As in the maxima representation case, for fixed sets $ZW_j f$, the remaining data $U_j^{zf} f$ and $S_J f$ are obtained by a linear operator, denoted by T_{zf}. The zero-crossings representation can also be written as:

$$R_z f = \left\{\{ZW_j f\}_{j=1}^{J}, T_{zf} f\right\}. \tag{12}$$

We have the following characterization of the reconstruction set.

Theorem 2 *Let $R_z f$ be a given multiscale zero-crossings representation. $h \in \Gamma(R_z f)$ if and only if*

$$T_{zf} h = T_{zf} f \tag{13}$$

$$t(k) \cdot W_j h(k) > 0. \tag{14}$$

$t(k)$ is the type of k and can be either $+1$ or -1. The last inequality should be satisfied for almost all k (if $W_j f(k) = 0$ then k is omitted).

As an immediate consequence of Theorem 2 we have:

Proposition 3 *Any multiscale zero-crossings representation is an AQLR.*

Moreover:

Theorem 3 *The wavelet zero-crossings representation is an inherently bounded AQLR.*

4 Nonuniqueness

A representation $Rf = \{Vf, Tf\}$ is said to be unique, if the reconstruction set $\Gamma(Rf)$ consists of exactly one element. We have the following uniqueness characterization for AQLR's.

Lemma 1 *Let $Rf = \{Vf, Tf\}$ be an AQLR. Then Rf is unique if and only if the kernel of the operator T is trivial, i.e. $\mathcal{N}T = \{0\}$.*

The proof is clear from topological considerations. Nevertheless, an elementary but constructive proof is given in [3].

This claim has some significant consequences. Using the above lemma, an algorithm which tests for uniqueness can be developed. One option is to derive it from a rank test of the operator T. Another, more ambitious, approach is to characterize, for a particular application, all sets Vf giving rise to a unique representation. Perhaps the most important consequence of Lemma 1 is the fact that uniqueness of the representation Rf is equivalent to uniqueness of the underlying irregular sampling Tf. In other words, in the unique case, all the information about the signal is already contained in Tf. Additional constraints $Af > a$ are redundant. On the other hand, from the signal compression, understanding and interpretation point of view, it seems to be desirable that a little information would be specified explicitly by Tf and as much as possible information about a signal should be described implicitly by $Af > a$. Therefore, in our opinion, the most important and interesting features of AQLR's appear in the nonunique case.

Using the previous lemma, we are able to show that:

Theorem 4 *A discrete dyadic wavelet maxima (zero-crossings) representation based on a discrete low pass filter $H(\omega)$ is given. If $H(\pi) = 0$, $J \geq 3$, and N is a multiple of 2^J then there exists a sequence f which has a nonunique maxima (zero-crossings) representation.*

Let us point out that, although, the hypothesis of the theorem may seem to be demanding, it is just a technical condition. Usually the number of levels, J, satisfies $J \geq 3$. In order to benefit from the fast wavelet transform N has to be a multiple of 2^J. Since $H(\omega)$ is a low pass filter, it is natural to assume that $| H(\omega) |$ reaches its minimum at π. If this minimum is nonzero, then essentially $S_J f$ contains all information about f and the remaining maxima (zero-crossings) information is redundant. Indeed, all filters used by Mallat, Zhong and many others fulfill the conditions of the theorem.

As a generic example of nonuniqueness the following sequence is proposed.

$$f(n) = \cos(2\pi \frac{n}{2^J}) \quad n = 0, 1, \ldots, N-1. \qquad (15)$$

Observe that the same sequence is proposed for all dyadic wavelet transforms and for both the maxima representation and the zero-crossings representation. For details of the proof see [3], for the specific example see [1, 2].

5 Stability

To address the stability issue, the standard approach is to introduce the notion of perturbation of the representation, and of the reconstruction set. In addition, distance measures between distinct representations and between different reconstruction sets should be defined. In general, this is not an easy task. Observe that Vf, Tf may have different sizes for different representations. Fortunately, for inherently bounded representations, the following characterization of BIBO (bounded input, bounded output) stability is easily verified.

Proposition 4 *Let $Rf_i = \{Vf_i, T_i f_i\}$ $(i = 1, 2)$ be inherently bounded AQLR's. Then for all $K_I > 0$ there exists K_O such that:*

$$\|T_i f_i\| \leq K_I \quad (i = 1, 2) \Rightarrow$$

$$\|x_1 - x_2\| \leq K_O \quad \forall x_i \in \Gamma(Rf_i)$$

Proof: This claim is an immediate consequence of the definition of an inherently bounded AQLR.

$$x_i \in \Gamma(Rf_i) \Rightarrow \|x_i\| \leq K \cdot \|T_i f_i\| \leq K \cdot K_I$$

$$\|x_1 - x_i\| \leq \|x_1\| + \|x_2\| \leq 2K \cdot K_I \qquad \Box$$

In many applications, the reasons for perturbations in a representation are arithmetic or quantization errors in a reconstruction algorithm. This kind of perturbations may change the continuous values of Tf but it preserves the discrete values of Vf. Therefore the perturbed representation, $(Rf)_p$, can be written as:

$$(Rf)_p = \{Vf, Tf + \Delta(Tf)\}. \qquad (16)$$

Let Γ_p be the corresponding reconstruction set. In general, we define the distance, d, between two reconstruction sets, Γ and Γ_p as:

$$d(\Gamma, \Gamma_p) \triangleq \sup\{\|\gamma - \gamma_p\| : \gamma \in \Gamma, \gamma_p \in \Gamma_p\}.$$

Observe, that for an inherently bounded AQLR, $d(\Gamma, \Gamma_p)$ is always finite. The measure of the perturbation in the reconstruction set is the difference between $d(\Gamma, \Gamma_p)$ and the size of Γ which is defined as follows:

$$s(\Gamma) \triangleq d(\Gamma, \Gamma) = \sup\{\|\gamma_1 - \gamma_2\| : \gamma_1, \gamma_2 \in \Gamma\}. \qquad (17)$$

$s(\Gamma)$ and $d(\Gamma, \Gamma_p)$ describe the largest possible Euclidian norm of a reconstruction error, from the original representation and from a perturbed one, respectively.

One remark is in order. In general, for an arbitrary $\Delta(Tf)$, the associated reconstruction set may be empty and then $d(\Gamma, \Gamma_p)$ would not be defined. In the sequel, it is assumed that this problem is treated by a reconstruction algorithm and hence $\Delta(Tf)$ yields a nonempty Γ_p. In this case, the following Lipschitz condition is satisfied.

Theorem 5 *For all inherently bounded AQLR, there exists $K > 0$ such that:*

$$d(\Gamma, \Gamma_p) \leq K \cdot \|\Delta(Tf)\| + s(\Gamma). \qquad (18)$$

Due to limited space, the detailed proof is omitted. The technique of the proof is borrowed from linear parametric programming. The first step is to observe that the distance between two reconstruction sets is given as the norm of the difference between two vertices, one from each set. Then by careful analysis of possible perturbation in these vertices, the claim of the theorem can be shown. The complete proof is given in [3]. Observe that the above result is global in the sense that as long as $\Delta(Tf)$ gives rise to a nonempty reconstruction set, the theorem holds regardless of the size of $\Delta(Tf)$.

6 A New Reconstruction Scheme

In a nonunique case, there are several ways to define a reconstruction algorithm. One can require to find all elements from the reconstruction set, sometimes it is desired to determine a smallest element satisfying a given representation. In this work, the reconstruction is defined as a procedure to find any element x belonging to the closure of the reconstruction set, $\overline{\Gamma}$. As mentioned earlier, we propose a reconstruction algorithm based on an appropriate potential function $v(x)$. This function should satisfy:

$$v(x) = 0 \quad \forall x \in \overline{\Gamma} \tag{19}$$

$$v(x) > 0 \quad \forall x \in \left(\overline{\Gamma}\right)^c. \tag{20}$$

where $\left(\overline{\Gamma}\right)^c$ denotes the complement of $\overline{\Gamma}$ in $\mathcal{L}$. Furthermore, it will be shown that the proposed $v(x)$ does not have any local extremum outside $\overline{\Gamma}$, i.e.

$$\|\nabla v(x)\| > 0 \quad \forall x \in \left(\overline{\Gamma}\right)^c. \tag{21}$$

$\nabla v(x)$ denotes the gradient of $v(x)$ with respect to x, namely it is a column vector of derivatives of v with respect to components of x. We will focus on the reconstruction algorithm based on the differential equation:

$$\dot{x}(t) = -\nabla\left(v\left(x(t)\right)\right) \tag{22}$$

whose analog hardware implementation gives rise to a very fast algorithm.

In this section, a general inherently bounded Adaptive Quasi Linear Representation (AQLR) is considered. By a standard procedure, the closure of the reconstruction set, $\overline{\Gamma}$, can be written as:

$$\overline{\Gamma} = \{x : Bx \geq b\} \tag{23}$$

for a given $p \times N$ matrix B and a p-dimensional vector b. The function $v(x)$ is derived from this representation in the subsequent way.

$$v(x) \triangleq \sum_{i=1}^{p} f(Bx - b)_i \tag{24}$$

where $(Bx - b)_i$ denotes the i-th component of the vector $Bx - b$. The function $f(\cdot)$ is defined by:

$$f(\xi) \triangleq \begin{cases} \xi^2 & \text{if } \xi < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Using the above definitions, it is easy to verify that indeed (19) and (20) hold.

Observe that $f(\xi)$ is continuously differentiable. Therefore $v(x)$ is continuous and continuously differentiable. The gradient of $v(x)$ is given by:

$$\nabla v(x) = 2B'Z(Bx - b)$$

where Z is a $p \times p$ diagonal matrix satisfying:

$$Z(i,i) = \begin{cases} 1 & \text{if } (Bx - b)_i < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Naturally, B' denotes the transpose of the matrix B.

The following theorem states that $v(x)$ does not have local extrema outside the set $\overline{\Gamma}$.

Theorem 6 *Let Γ be nonempty. Then $\nabla v(x) = 0$ if and only if $x \in \overline{\Gamma}$.*

The proof is given in [3]. The idea is to represent Zb as a sum of two elements: one from the range of Zb and its orthogonal complement. Then applying Farkas' Lemma to the fact that $\overline{\Gamma}$ is nonempty verifies the claim of the theorem.

In view of these considerations, a reconstruction scheme can be implemented as:

$$\arg\min \{v(x) : x \in \mathcal{L}\}. \tag{25}$$

The minimization is significantly facilitated by the property that local extrema of $v(x)$ appear only in $\overline{\Gamma}$. We are going to focus on the algorithm based on the differential equation (22). The desired property is that for all $x(0)$, $x(t)$ will approach the set $\overline{\Gamma}$ as $t \rightarrow \infty$. The convergence result is stated as follows.

Theorem 7 *Let $\overline{\Gamma}$ be the closure of the reconstruction set of the inherently bounded AQLR. Then for all $x(0)$, the solution of*

$$\dot{x}(t) = -\nabla\left(v\left(x(t)\right)\right). \tag{26}$$

will approach $\overline{\Gamma}$ as $t \rightarrow \infty$.

The detailed proof of Theorem 7 is given in [3].

The complexity of calculating the gradient of v, in the case of the wavelet-based maxima (zero-crossings) representation, is of the order of complexity of the wavelet transform itself. Therefore, in this case, applying an analog-based integration can lead to a very efficient implementation.

7 Conclusions

Perhaps the most important outcome of this work is to show feasibility and capability of discrete analysis. In general, the discrete approach described here may be applied to a variety of representations and reconstruction algorithms, providing new insights into their properties. We believe that, even for complex algorithms, testing for uniqueness and computing a precise reconstruction set, even for a few examples, is worth the effort.

As mentioned earlier, many important and interesting features of the multiscale edge representation appear in the nonunique case. However, the core of theoretical studies has been developed in the framework of unique representations. In our opinion, the need to develop more analytical tools and applications for nonunique representations is apparent.

As a first step in the undergoing research, this work dealt only with one dimensional signals. The reason is twofold: firstly, we thought that in the simpler case the basic properties would be better recognizable, secondly, the one dimensional multiscale edge representation has its own variety of applications. One of the most promising application areas is speech analysis, for example, pitch detection [5] or modeling signal transformations in auditory nervous system [10]. On the other hand, up to this point, the vast majority of multiscale edge representations has been implemented in computer vision. Therefore, it is advisable to extend these results for two dimensional signals. Surprisingly, there is an essential difference between maxima and zero-crossings representations. A two dimensional multiscale zero-crossings representation can easily be cast into the structure of the inherently bounded AQLR, thus the related results are valid in this case. However, a two dimensional maxima representation appears to have a different structure. In order to proceed with a similar analysis one either has to extend the framework of the AQLR, or to change the definition of the two dimensional maxima representation to match the structure of the AQLR. We are in the process of deciding which choice is more suitable for analysis and applications.

Summarizing, the described theoretical results about uniqueness and stability are new. In our opinion, the most significant contribution of this work is to create a framework to define, analyze, and reconstruct a wide family of representations. Important examples are generalizations of a basic maxima representation obtained by using only a subset of local extreme points. Their properties are the subject of the undergoing research.

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