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Two-Dimensional Signal Deconvolution Using Multiple Sensors *

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Abstract

We consider the two dimensional Analytic Bezout Equation (ABE) and investigate the properties of a particular solution, based on certain conditions imposed on the convolution kernels. We propose the use of approximate deconvolution kernels, prove compactness of their support, and discuss various design problems pertaining to the question of convergence of the proposed deconvolution kernels. We point out the fact that these approximate deconvolution kernels exhibit a strong asymmetry in the FT domain, and propose solutions to this problem. Simulation results are presented in order to demonstrate the gain in bandwidth, the asymmetry of the FT of the overall system, and the effectiveness of the proposed solutions to compensate for this asymmetry.

1 Introduction

Signal deconvolution is a fundamental problem related to a variety of scientific and engineering disciplines. The traditional problem formulation can be stated as follows: we observe the output of a Linear Time Invariant system modelled by a convolution operator with known kernel (or convoluter) and wish to synthesize the input signal based on output observations. This is generally an ill-posed problem. An alternative approach is to use *a family* of suitably chosen Linear Time Invariant convolution operators and attempt to reconstruct the common input signal by combining the outputs of all available devices. The motivation here stems from the fact that multiple operators are indeed necessary for the deconvolution problem to be well posed [1]. The specific application we have in mind is deconvolution for electro-optic imaging devices (Imaging Detector Arrays) [1,10,6].

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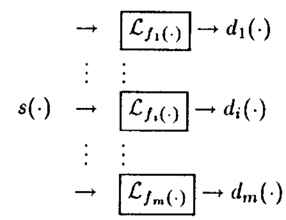

Figure 1: Multiple convolutional operators operating on a single input

Consider the system of figure 1. The $f_i's$ are distributions of compact support defined over $\mathcal{R}^2$ and $\mathcal{L}_{f_i}$ denotes convolution with kernel f_i. The natural question that comes up is: what is the minimum possible m and what conditions should the f_i's satisfy so that we can uniquely determine $s(\cdot)$ from the $d_i(\cdot)'s$? We are specifically interested in obtaining *linear* estimates of the input signal based on output observations from the bank of available devices. Mathematically the problem can be formulated as a convolution equation. We are looking for a family of *deconvolvers* $h_i(\cdot)$, $i = 1, \cdots, m$ such that:

$$f_1 * h_1 + \cdots + f_m * h_m = \delta \qquad (1)$$

Here, δ denotes the unit impulse located at the time origin. Alternatively, we need a family of entire analytic functions $\widehat{h}_i(\cdot)$, $i = 1, \cdots, m$ such that

$$\widehat{f}_1\widehat{h}_1 + \cdots + \widehat{f}_m\widehat{h}_m = 1 \qquad (2)$$

Here $\widehat{\ }$ denotes Fourier Transform. The later equation is known as the Analytic Bezout Equation (ABE). It is a well known fact that the existence of a family of deconvolvers, $\{\widehat{h}_1, \cdots, \widehat{h}_m\}$ that solves the Bezout Equation is completely equivalent to a coprimeness condition on the part of the f_i's.

2 Existence and construction of deconvolvers of compact support

Let $\mathcal{E}'_{\mathcal{R}^2}$ denote the space of all distributions of compact support defined over $\mathcal{R}^2$. Let $\widehat{\mathcal{E}}'_{\mathcal{R}^2}$ denote the Paley-Wiener space. The mapping $\mathcal{E}'_{\mathcal{R}^2} \mapsto \widehat{\mathcal{E}}'_{\mathcal{R}^2}$ given by $f \mapsto \widehat{f}$, where $\widehat{\ }$ denotes

Fourier transform, for all $f \in \mathcal{E}'_{\mathcal{R}^2}$, is 1-1 and onto the Paley-Wiener Space $\widehat{\mathcal{E}}'_{\mathcal{R}^2}$. Therefore, we can work with entire analytic functions and equation (2) instead of distributions of compact support and equation (1). For convenience we drop the index $\mathcal{R}^2$.

Theorem 1 *[9] There exists a family of functions $\{\widehat{h}_1, \cdots, \widehat{h}_m\}$ in $\widehat{\mathcal{E}}'$ that solves the Bezout Equation iff the family of entire functions $\{\widehat{f}_1 \cdots, \widehat{f}_m\}$ in $\widehat{\mathcal{E}}'$ is strongly coprime, i.e.* **iff**: *$\sum_{j=1}^{m} |\widehat{f}_j(\omega)|^2 \geq e^{-cp(\omega)}, \forall \omega \in C^2$, for some constant c. Here, $p(\omega) = |Im\omega| + log(1 + |\omega|)$.*

Definition 1 *Let K be a compact subset of $\mathcal{R}^2$. Define the* **supporting function** *of K as follows:*

$$H_K(\xi) \triangleq max\{x \cdot \xi | x \in K\} \qquad (3)$$

where $\cdot$ denotes inner product and $\xi \in \mathcal{R}^2$.

Consider a family of 2 distributions $\{f_1, f_2\}$ of compact support in $\mathcal{R}^2$. Let H_1 denote the supporting function of the convex hull of the union of the support sets of f_1, f_2. It can be shown that H_1 can be written as

$$H_1(\theta) = max_{1 \leq j \leq 2} max\{x \cdot \theta \ : x \in sprt f_j\}$$

for all $\theta \in \mathcal{R}^2$.

Definition 2 *A family of 2 distributions $\{f_1, f_2\}$ of compact support in $\mathcal{R}^2$ is* **well behaved** *if there exist positive constants A, B, N, K, C and a supporting function H_0, such that $0 \leq H_0 \leq H_1$, such that the common zero set, $\mathcal{Z}$, of the functions $\{\widehat{f}_1, \widehat{f}_2\}$ is almost real i.e. $\forall \omega \in \mathcal{Z} : |Im\omega| \leq C\, log(2 + |\omega|)$, and the number of zeros in $\mathcal{Z}$ included in an open ball of radius r grows like r^A, $n(\mathcal{Z}, r) = O(r^A)$, and denoting*

$$|\widehat{f}(z)| \triangleq \left[\Sigma_{i=1}^{2} |\widehat{f}_i(z)|^2\right]^{1/2} \quad (4)$$

the following inequality holds

$$|\widehat{f}(z)| \geq \frac{Bd(z, \mathcal{Z})^K e^{H_o(Imz)}}{(1 + |z|)^N} \qquad (5)$$

where $d(z, \mathcal{Z})$ is the minimum of 1 and the Euclidean distance of the point z from the set $\mathcal{Z}$. It can be shown that under these conditions the set $\mathcal{Z}$ is discrete, i.e. the points $\zeta \in \mathcal{Z}$ are isolated.

Definition 3 *A well-behaved family $\{f_1, f_2\}$* **is very well behaved** *if there exist constants $M, C_1 > O$ such that $\forall \zeta \in \mathcal{Z}$ we have that:*

$$|J(\zeta)| \triangleq |det[\frac{\partial \widehat{f}_j}{\partial z_i}(\zeta)]_{ij}| \geq C_1(1 + |\zeta|)^{-M} \quad (6)$$

This last condition guarantees that the points in $\mathcal{Z}$ (the set of common zeros of the family $\{\widehat{f}_1, \widehat{f}_2\}$) are **well separated**, i.e. there exist constants $M', C_2 > 0$ such that for any $\zeta \in \mathcal{Z}$ there exists $r = r(\zeta)$ such that

$$r(\zeta) \geq \frac{C_2}{(1 + |\zeta|)^{M'}}$$

and such that the open ball $B_r(\zeta)$ contains no other points in $\mathcal{Z}$.

Theorem 2 *[5, p.57] Let $\{f_1, f_2, f_3\}$ be a strongly coprime family of compactly supported distributions over $\mathcal{R}^2$. Suppose that the subfamily $\{f_1, f_2\}$ is very well behaved. Suppose f_3 is the "best" kernel in the sense that it has the smallest support of all three. Let H_0, H_1 be as in definition 2 for the subfamily $\{f_1, f_2\}$. Let sprt f denote the support set of the distribution f, and for all $\theta \in \mathcal{R}^2$ define*

$$H_2(\theta) = max_{1 \leq j \leq 3} max\{x \cdot \theta \ : x \in sprt f_j\}$$

and suppose $H_2 \leq 2H_1$. Furthermore suppose $\exists\, r_o > 0$ such that $r_o|\theta| \leq 4H_o(\theta) - 2H_1(\theta) - H_2(\theta)$ (these conditions control the support of f_3 vs. the support of f_1, f_2). Then for any $u \in C_o^\infty(\mathcal{R}^2)$ compactly supported and of sufficiently narrow support, sprt $u \subseteq \{x \in \mathcal{R}^n : |x| \leq r_0\}$, $\widehat{u}(z)$ can be written as

$$\widehat{u}(z) = \sum_{\zeta \in \mathcal{Z}} \frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)} D(z, \zeta) \qquad (7)$$

where $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2)$, both in C^2, and

$$D(z, \zeta) \triangleq \begin{vmatrix} g_1^1(z,\zeta) & g_1^2(z,\zeta) & g_1^3(z,\zeta) \\ g_2^1(z,\zeta) & g_2^2(z,\zeta) & g_2^3(z,\zeta) \\ \widehat{f}_1(z) & \widehat{f}_2(z) & \widehat{f}_3(z) \end{vmatrix} \qquad (8)$$

$$g_1^i(z, \zeta) \triangleq \frac{\widehat{f}_i(z_1, \zeta_2) - \widehat{f}_i(\zeta_1, \zeta_2)}{z_1 - \zeta_1} \qquad (9)$$

$$g_2^i(z, \zeta) \triangleq \frac{\widehat{f}_i(z_1, z_2) - \widehat{f}_i(z_1, \zeta_2)}{z_2 - \zeta_2} \qquad (10)$$

Here, $J(\zeta) = det(M(z))|_{z=\zeta}$, where the Jacobian matrix M(z) is defined as

$$M(z) \triangleq \begin{bmatrix} \frac{\partial \widehat{f}_1}{\partial z_1} & \frac{\partial \widehat{f}_2}{\partial z_1} \\ \frac{\partial \widehat{f}_1}{\partial z_2} & \frac{\partial \widehat{f}_2}{\partial z_2} \end{bmatrix} \qquad (11)$$

and

$$\mathcal{Z} = \{z \in C^2 : \widehat{f}_1(z) = \widehat{f}_2(z) = 0\} \qquad (12)$$

The significance of Theorem 2 can be demonstrated by a simple manipulation of equation (7).

$$\begin{aligned} D(z, \zeta) = \\ [g_1^2(z,\zeta)g_2^3(z,\zeta) - g_1^3(z,\zeta)g_2^2(z,\zeta)]\, \widehat{f}_1(z) \\ + [g_1^3(z,\zeta)g_2^1(z,\zeta) - g_1^1(z,\zeta)g_2^3(z,\zeta)]\, \widehat{f}_2(z) \\ + [g_1^1(z,\zeta)g_2^2(z,\zeta) - g_1^2(z,\zeta)g_2^1(z,\zeta)]\, \widehat{f}_3(z) = \\ \widehat{\phi}_1(z,\zeta)\widehat{f}_1(z) + \widehat{\phi}_2(z,\zeta)\widehat{f}_2(z) + \widehat{\phi}_3(z,\zeta)\widehat{f}_3(z) \end{aligned} \qquad (13)$$

Thus equation (7) can be rewritten as

$$\widehat{u}(z)=\widehat{h_1}(z)\widehat{f_1}(z)+\widehat{h_2}(z)\widehat{f_2}(z)+\widehat{h_3}(z)\widehat{f_3}(z) \tag{14}$$

With

$$\begin{aligned}\widehat{h}_1(z) &\triangleq \textstyle\sum_{\zeta\in\mathcal{Z}} \frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)}\widehat{\phi}_1(z,\zeta)\\ \widehat{h}_2(z) &\triangleq \textstyle\sum_{\zeta\in\mathcal{Z}} \frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)}\widehat{\phi}_2(z,\zeta) \qquad (15)\\ \widehat{h}_3(z) &\triangleq \textstyle\sum_{\zeta\in\mathcal{Z}} \frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)}\widehat{\phi}_3(z,\zeta)\end{aligned}$$

And since $u(x)$ is of sufficiently small support (we can certainly shrink the support of u below r_o) then $\widehat{u}(z) \cong 1$ and $\{\widehat{h}_1(z),\widehat{h}_2(z),\widehat{h}_3(z)\}$ give an approximate solution to the ABE. Next we will show that there exist unique distributions of compact support, $\{h_1(t),h_2(t),h_3(t)\}$, with corresponding FT $\{\widehat{h}_1(z),\widehat{h}_2(z),\widehat{h}_3(z)\}$. Let us consider $\widehat{h}_1(z)$. The development for the other two follows along the same lines. It suffices to show that every term of the sum over $\zeta\in\mathcal{Z}$ is the Fourier transform of a distribution of compact support and obtain an upper bound on its support *which is independent of* ζ. This is crucial. Fix ζ and consider the following function (which is analytic in z)

$$\frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)}\left[g_1^2(z,\zeta)g_2^3(z,\zeta)-g_1^3(z,\zeta)g_2^2(z,\zeta)\right] \tag{16}$$

For ζ fixed, the first factor is just a scaling constant. By [5, Lemma 1 p.54] $g_k^i(z,\zeta)$ is the Fourier transform of a distribution of compact support. Furthermore *ch sprt* $FT^{-1}\{g_k^i(z,\zeta)\}\subseteq$ *ch sprt* f_i. Here ch denotes convex hull of a set. Therefore, every term of the sum over $\zeta\in\mathcal{Z}$ is the Fourier transform of a distribution of compact support, whose support is bounded independently of ζ. Hence the compactness of support of the proposed deconvolution kernels follows.

It has to be emphasized that there are two levels of approximation here. First, we generally *choose* u to be different from δ for reasons that are going to be discussed in section 3. This results in a family of deconvolvers that approximate the exact deconvolvers. Second, we further approximate these deconvolvers by truncating the corresponding sums. Let us call the deconvolvers of the first level of approximation the *intended* ones, and the deconvolvers of the second level of approximation the *realizable* ones. These realizable deconvolvers are going to be compactly supported by virtue of the fact that *every term* of the sums over $\zeta\in\mathcal{Z}$ in equation (15) is the FT of a distribution of compact support whose support can be bounded independently of ζ.

The compactness of support of these realizable deconvolution kernels is their most desirable feature when compared to Wiener deconvolvers. The reason is that they can be implemented *exactly* with finite delay (here exactly refers to the fact that there is no need to truncate their duration; sampling and finite word length arithmetic can be controlled to meet the design goals [13]). Wiener deconvolvers have been shown to be optimal in the presence of noise, under certain reasonably plausible assumptions (namely that the noise is additive, with one sample function of a Wide Sense Stationary, zero mean random process added to the output of each convolver (sensor), and noise random processes corresponding to different sensors are independent of each other, while all noise random processes are independent of the input signal) [1,6]. Numerically, the proposed deconvolvers are very close to the Wiener Deconvolvers in the Fourier transform domain, except for a certain degree of rounding up of very sharp peaks present in the Wiener deconvolvers (we attribute this to the fact that the proposed deconvolvers are analytic, and, therefore, cannot follow very sharp peaks exactly). Hence the behaviour of the proposed deconvolvers in the presence of noise is expected to be very close to optimal.

Another important issue is the robustness of the proposed deconvolution scheme under small perturbations of the convolution kernels (continuity of the overall operator with respect to the actual analog approximations of the $f_i's$) and/or the actual digital approximations of the deconvolvers. This issue has been discussed in [6]. The findings suggest that the proposed scheme is fairly robust under such approximations.

In the section that follows we consider a specific two-dimensional example. The analysis for the general case (of having arbitrary very well behaved kernels that constitute part of a strongly coprime family) follows along the same lines, although some computational issues can become more complicated.

3 Two-Dimensional Example

Let χ_K denote the characteristic function of the compact set $K\subset\mathcal{R}^n$ and consider the following family of convolution kernels:

$$\begin{aligned}f_1(t_1,t_2) &= \chi_{[-\sqrt{3},\sqrt{3}]\times[-\sqrt{3},\sqrt{3}]}(t_1,t_2)\\ f_2(t_1,t_2) &= \chi_{[-\sqrt{2},\sqrt{2}]\times[-\sqrt{2},\sqrt{2}]}(t_1,t_2)\\ f_3(t_1,t_2) &= \chi_{[-1,1]\times[-1,1]}(t_1,t_2)\end{aligned} \tag{17}$$

with Fourier transforms given by

$$\begin{aligned}\widehat{f}_1(z_1,z_2) &= \tfrac{4}{z_1z_2}\sin(\sqrt{3}z_1)\sin(\sqrt{3}z_2)\\ \widehat{f}_2(z_1,z_2) &= \tfrac{4}{z_1z_2}\sin(\sqrt{2}z_1)\sin(\sqrt{2}z_2)\\ \widehat{f}_3(z_1,z_2) &= \tfrac{4}{z_1z_2}\sin(z_1)\sin(z_2)\end{aligned} \tag{18}$$

Then it is easy to verify that $\{f_1,f_2,f_3\}$ satisfy all conditions of theorem 2. The Fourier transforms of the resulting deconvolvers are given by the infinite sums

$$\widehat{h}_i(z_1,z_2) = \sum_{\zeta\in\mathcal{Z}} \frac{\widehat{u}(\zeta)}{J(\zeta)\widehat{f}_3(\zeta)} \cdot \frac{C_i(z,\zeta)}{(z_1-\zeta_1)(z_2-\zeta_2)} \tag{19}$$

with

$$C_1(z,\zeta) \triangleq \widehat{f}_2(z_1,\zeta_2)\left[\widehat{f}_3(z_1,z_2)-\widehat{f}_3(\zeta_1,\zeta_2)\right] -\widehat{f}_2(z_1,z_2)\left[\widehat{f}_3(z_1,\zeta_2)-\widehat{f}_3(\zeta_1,\zeta_2)\right] \tag{20}$$

$$C_2(z,\zeta) \triangleq \widehat{f}_1(z_1,z_2)\left[\widehat{f}_3(z_1,\zeta_2)-\widehat{f}_3(\zeta_1,\zeta_2)\right] -\widehat{f}_1(z_1,\zeta_2)\left[\widehat{f}_3(\zeta_1,\zeta_2)-\widehat{f}_3(z_1,z_2)\right] \tag{21}$$

$$C_3(z,\zeta) \triangleq \widehat{f}_1(z_1,\zeta_2)\widehat{f}_2(z_1,z_2)-\widehat{f}_1(z_1,z_2)\widehat{f}_2(z_1,\zeta_2) \tag{22}$$

Definition 4 *The function* $\mathcal{F}(\cdot,\cdot)$ *is* **symmetric** *iff*

$\mathcal{F}(z_1,z_2)=\mathcal{F}(-z_1,z_2),\ \forall z_1,z_2\in\mathcal{C}^2$
and
$\mathcal{F}(z_1,z_2)=\mathcal{F}(z_1,-z_2),\ \forall z_1,z_2\in\mathcal{C}^2.$

Definition 5 *The function* $\mathcal{F}(\cdot,\cdot)$ *is* $\frac{\pi}{2}$ **rotation-invariant** *(* $\frac{\pi}{2}-ri$ *) iff* $\mathcal{F}(z_1,z_2)=\mathcal{F}(z_2,z_1),\ \forall z_1,z_2\in\mathcal{C}^2.$

Notice that $\widehat{f}_1$, $\widehat{f}_2$, $\widehat{f}_3$ are all $\frac{\pi}{2}$-ri. Nevertheless the C_i's *are not*; for example $C_1(z_1,z_2)\neq C_1(z_2,z_1)$. Hence, in general, every *finite* approximation to $\widehat{h}_1$ will not be $\frac{\pi}{2}$-ri. In the limiting case we expect this bias to die out because of cancellations. Similar remarks hold for $\widehat{h}_2,\widehat{h}_3$. This fact will prove annoying for applications.

4 Windowing and Averaging

Our goal is the pointwise evaluation of the FT of the intended deconvolution kernels over a suitably chosen finite grid. Clearly this involves the pointwise (in z) computation of an infinite sum. Therefore, we will have to truncate this sum at some point. Here, we must strike a balance between computational feasibility and noise averaging on one hand, and quality of deconvolution on the other. In principle $u(t)$ can by anything as long as it is sufficiently differentiable and compactly supported. Noise considerations dictate a *smooth* choice of u(t) which in turn implies a fast decay of $\widehat{u}(z)$ at infinity. The choice of u strongly affects the convergence of the realizable deconvolvers to the intended deconvolvers (because the smoother u is, the faster the decay of $\widehat{u}$ at infinity, and, therefore, the faster the convergence). Therefore u should be chosen (among other things) to be smooth enough to guarantee sufficiently fast convergence and good noise rejection of the overall design. If these issues were of no concern then we would like u(t) to be as close to $\delta(t)$ as possible, or, equivalently, $\widehat{u}(z)$ to be as close to unity as possible, in order to achieve as good reconstruction of the original signal as possible. Thus we have to accommodate conflicting interests. It is not clear what is a proper choice for $\widehat{u}$. Various different candidate functions were considered, and simulations were carried out in order to bring out the differences in performance between them in achieving the stated requirements. These simulations indicated that the following family of functions is a good compromise, especially because it seemed to maximize the convergence rate.

$$\widehat{u}(z) = \left(\prod_{i=1}^{2}\frac{\sin(\frac{\epsilon_i}{N}z_i)}{\frac{\epsilon_i}{N}z_i}\right)^N \cdot p_r(z) \tag{23}$$

where N is a small positive integer and ϵ_1, ϵ_2 are small positive reals. The function $p_r(z)$ is defined as follows:

$$p_r(z) = \begin{cases} 1, & |z_i|\leq r\ ,\ i=1,2 \\ 0, & elsewhere \end{cases} \tag{24}$$

The first factor is a two dimensional sinc-like function. The parameter r (forced cutoff in rads/sec) is to be chosen sufficiently large to include all main features of the first factor (the main lobe and the principal sidelobes at least) , while keeping the size of the computation reasonable. This family of functions has several nice properties that allow for trade-off between the design goals stated above. Simulation results are given at the end of this section . For the case $\epsilon_1=\epsilon_2=\epsilon$ the transfer function of the overall system (convolvers followed by realizable deconvolvers, where the sums run over the 3200 zeros that are closest to the origin) exhibits a high degree of energy concentration along a ribbon-like neighborhood of the z_2 axis, while amplitudes everywhere else are attenuated by at least an order of magnitude. This spurious asymmetry has a profound effect on the transfer function of the overall system, even at frequencies in the vicinity of the origin. There exists no a priori reason for the appearance of such a bias (for the particular example is symmetric and $\frac{\pi}{2}$-ri) but rather the cause can be traced back to a somewhat arbitrary choice between two distinct possibilities in writing down interpolation formulas. Before we discuss this very important point let us give a partial "a posteriori" solution: if the transforms of the convolvers are symmetric and $\frac{\pi}{2}$-rotation invariant then a simple solution is the following.

Let $\widehat{h}_{i,n}(z_1,z_2)$, $i=1,2,3$ denote the obtained approximations of the Fourier transforms of the intended deconvolution kernels, where n denotes the cardinality of the subset of $\mathcal{Z}$ over which we sum. Next for $i=1,2,3$ define

$$\widehat{h}^L_{i,n}(z_1,z_2) \triangleq \widehat{h}_{i,n}(z_2,z_1) \tag{25}$$

and

$$\widehat{h}^{*}_{i,n}(z_1,z_2) \triangleq \frac{\widehat{h}_{i,n}(z_1,z_2)+\widehat{h}^{L}_{i,n}(z_1,z_2)}{2} \quad (26)$$

By definition $\widehat{h}^{*}_{i,n}(\cdot,\cdot)$ is $\frac{\pi}{2}$ rotation invariant. Let $\widetilde{h}_i$, $i = 1,2,3$ denote the Fourier transforms of the intended deconvolution kernels. Then by theorem 2, for $i = 1,2,3$, we have

$$\widehat{h}_{i,n} \longrightarrow \widetilde{h}_i, \ as\ n \longrightarrow \infty \quad (27)$$

Thus, by symmetry and $\frac{\pi}{2}$ rotation invariance of the solution:

$$\widehat{h}^{L}_{i,n} \longrightarrow \widetilde{h}^{L}_i \equiv \widetilde{h}_i, \ as\ n \longrightarrow \infty \quad (28)$$

Hence the same is true for their average, i.e. the family $\{\widehat{h}^{*}_{i,n};\ i = 1,2,3\}$ constitutes a converging solution. Simulation results for this averaged solution are given at the end of this section.

We now turn to the general case where the Fourier transforms of the convolvers are not symmetric and $\frac{\pi}{2}$ - rotation invariant. In this case it is not necessarily true that $\widetilde{h}^{L}_i \equiv \widetilde{h}_i$ and the remedy above fails. In fact *we expect* the intended solutions to be asymmetric and/or biased. Nevertheless we have to account for spurious responses introduced by the need to come up with a finite computation because otherwise the results will be severely distorted. We now investigate the cause of these problems and proceed to propose a definitive solution. Consider the first column of the determinant involved in the interpolation formula (7) of Theorem 2. The idea is to write [5]

$$\widehat{f}_1(z_1,z_2) - \widehat{f}_1(\zeta_1,\zeta_2) = (z_1-\zeta_1)\cdot g^1_1(z,\zeta) + (z_2-\zeta_2)\cdot g^1_2(z,\zeta) \quad (29)$$

Quite clearly this can also be achieved via

$$\widehat{f}_1(z_1,z_2) - \widehat{f}_1(\zeta_1,\zeta_2) = (z_1-\zeta_1)\cdot \bar{g}^1_1(z,\zeta) + (z_2-\zeta_2)\cdot \bar{g}^1_2(z,\zeta) \quad (30)$$

Where $\bar{g}^1_1(z,\zeta)$ and $\bar{g}^1_2(z,\zeta)$ are defined as follows

$$\bar{g}^1_1(z,\zeta) \triangleq \frac{\widehat{f}_1(z_1,z_2)-\widehat{f}_1(\zeta_1,z_2)}{z_1-\zeta_1} \quad (31)$$

$$\bar{g}^1_2(z,\zeta) \triangleq \frac{\widehat{f}_1(\zeta_1,z_2)-\widehat{f}_1(\zeta_1,\zeta_2)}{z_2-\zeta_2} \quad (32)$$

There is no *a priori* reason for choosing any particular pair of analytic functions; either will do. Nevertheless *some* choice has to be made. In the limit this choice makes no difference, but for finite n, since *either* expansion pair of analytic functions is biased, the overall system response is biased towards one of the two frequency variables. Therefore we can define

$$\bar{D}(z,\zeta) = \begin{vmatrix} \bar{g}^1_1(z,\zeta) & \bar{g}^2_1(z,\zeta) & \bar{g}^3_1(z,\zeta) \\ \bar{g}^1_2(z,\zeta) & \bar{g}^2_2(z,\zeta) & \bar{g}^3_2(z,\zeta) \\ \widehat{f}_1(z) & \widehat{f}_2(z) & \widehat{f}_3(z) \end{vmatrix} \quad (33)$$

with $\bar{g}^2_1$, $\bar{g}^2_2$, $\bar{g}^3_1$, $\bar{g}^3_2$ defined as follows

$$\bar{g}^2_1(z,\zeta) = \frac{\widehat{f}_2(z_1,z_2)-\widehat{f}_2(\zeta_1,z_2)}{z_1-\zeta_1} \quad (34)$$

$$\bar{g}^2_2(z,\zeta) = \frac{\widehat{f}_2(\zeta_1,z_2)-\widehat{f}_2(\zeta_1,\zeta_2)}{z_2-\zeta_2} \quad (35)$$

$$\bar{g}^3_1(z,\zeta) = \frac{\widehat{f}_3(z_1,z_2)-\widehat{f}_3(\zeta_1,z_2)}{z_1-\zeta_1} \quad (36)$$

$$\bar{g}^3_2(z,\zeta) = \frac{\widehat{f}_3(\zeta_1,z_2)-\widehat{f}_3(\zeta_1,\zeta_2)}{z_2-\zeta_2} \quad (37)$$

So we need to use equation (7) to obtain two sets of solutions: one using the original D, and one using $\bar{D}$ in place of D. Again since both solutions converge to the intended deconvolvers as $n \longrightarrow \infty$ the same is true for their average. Furthermore, the bias is canceled out and does not appear in the overall system spectrum. Simulation results are presented in the sequence of figures that follows. The sums are taken over the 3200 zeros which are located closest to the origin. A frequency step of 0.1718 rads/sec and a frequency resolution of 256×256 points is adopted throughout the whole sequence of simulations. The magnitude of the Fourier transform of the third convolution kernel (best one) is depicted in figure 2. The bandwidth of this kernel is the available bandwidth *before any attempt is made to deconvolve the common input signal.* It is given here for comparison purposes. The magnitude of the Fourier transform of the overall system (i.e. bank of convolvers followed by bank of realizable deconvolvers, whose outputs are summed up to produce the overall system output) using the $\widehat{u}(z)$ given by equation (23) with parameters $\epsilon_1 = \epsilon_2 = \epsilon = 0.1$, $N = 3$ is depicted in figure 3. Figure 4 depicts the magnitude of the Fourier transform of the overall system using the $\widehat{u}(z)$ given by equation (23) with parameters $\epsilon_1 = 0.1$, $\epsilon_2 = 0.5$, $N = 3$. Finally, figure 5 depicts the magnitude of the Fourier transform of the overall system using frequency averaging of the resulting deconvolution kernels which were used in the configuration whose FT is depicted in figure 4.

4 Conclusions

We have employed recent results of analysis in several complex variables to come up with a set of *compactly supported* approximate deconvolution kernels for the reconstruction of a two dimensional signal based on multiple linearly degraded versions of the signal with a family of kernels that satisfy suitable technical conditions. We have discussed the question of convergence of the proposed deconvolution kernels, as it relates to various other design considerations. A spurious asymmetry in the response of the overall system has been pointed out, and means to compensate for it have been given.

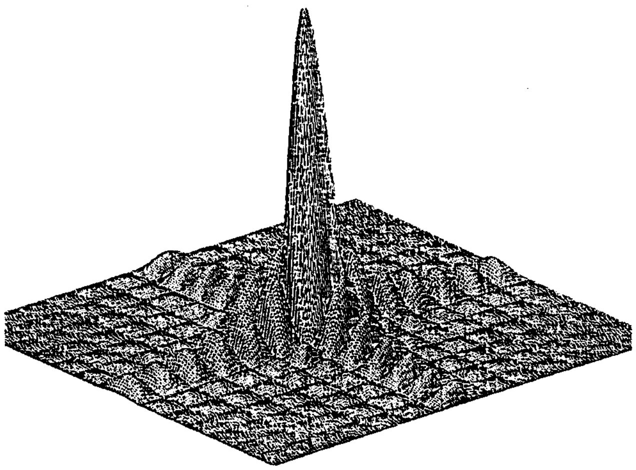

Figure 2: Magnitude of FT of convolver 3 (best one)

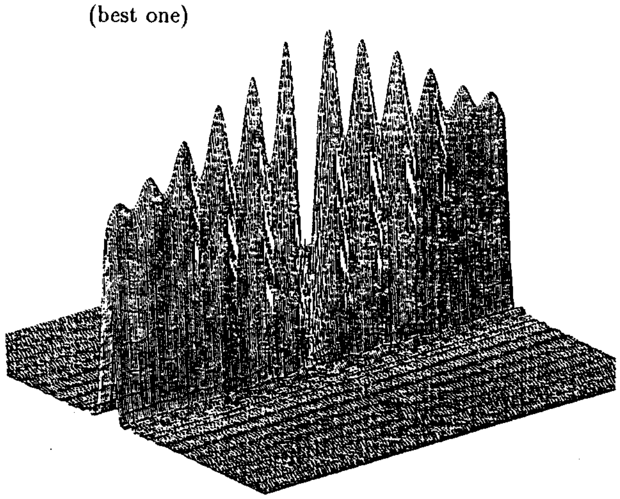

Figure 3: Magnitude of FT of overall system, $\epsilon = 0.1$, $N = 3$

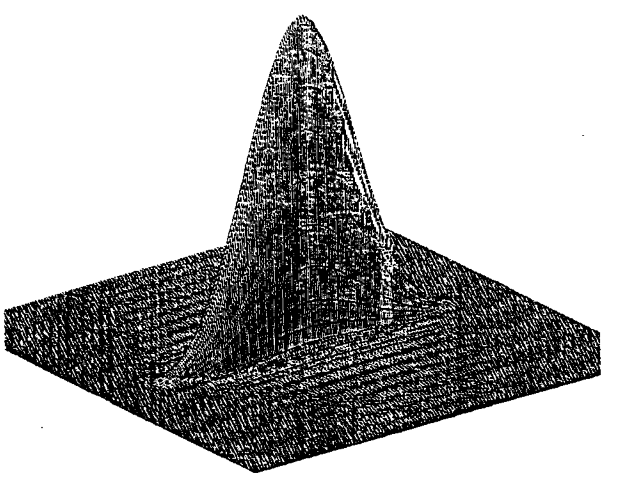

Figure 4: Magnitude of FT of overall system, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.5$, $N = 3$

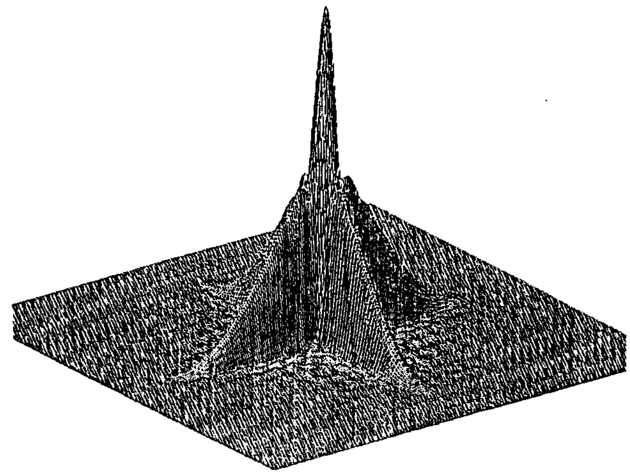

Figure 5: Magnitude of FT of overall system, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.5$, $N = 3$, *averaged*

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