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Robustness Issues in Boundary Feedback of Flexible Structures[1]

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Abstract

Transfer function models for basic structural elements with boundary control reveal certain inherent properties relevant to questions of control system realization, design, and analysis of robustness. The transfer functions involved are *not* strictly proper, may be nonminimum phase (except for the special case of colocated actuation and sensing), and often have large number of poles on the imaginary axis.

In this paper we consider these basic questions in terms of modeling and control system design for robustness. We investigate the application of algebraic methods for computing stabilizing control based on certain exact, irrational transfer functions. We consider boundary control of the wave equation and suggest extension of the method to more general problems. Special attention is given to implementation issues associated with a class of infinite dimensional control laws of the type described by Baras.

1.0 Introduction

Classical methods in structural analysis exploit frequency response modeling roughly for the purpose of extending static analysis techniques to dynamic problems. Application of several classical methods including "mechanical impedance" and "dynamic stiffness" methods to space structure modeling is considered in Pichè [1] (see also [2] for a classical treatment). In Poelaert [3] these methods are used to develop procedures for computation of "exact" modal frequencies. This is desirable primarily because of the limitations of approximation by finite element methods for structural modeling involving a large number of significant modal frequencies.

From the point of view of control system design and analysis it is known that transfer function analysis can be limited because of possible loss of observability and/or controlability so that certain critical modal responses may not be evident from the transfer function. The modern algebraic approach to control system design, which is based on transfer function models, exploits special algebraic constructions (like coprime factorizations) to delineate the scope of feedback control options. The key is a delicate interplay between the algebraic constructions offered by transfer function analysis and the internal representations offered by state-space theory.

The classical frequency domain treatment of feedback control provides a natural setting for quantifying robustness attributes of a chosen feedback control. In this paper we study the application of transfer function methods for computing stabilizing control laws with boundary feedback and observation of certain simple distributed structures. We address a robustness question (which has also been addressed in [4]) in this classical setting and demonstrate that although certain sensitivities are evident that these problems occur largely when material damping is ignored in the distributed model. We suggest that alternate methods for controller realization are possible for certain irrational transfer functions arising in boundary control of elastic structures. We give illustrations suggesting that the realizations are both practical and efficient when a digital computer is available for realtime control.

Recent developments in the algebraic theory of control systems has provided answers to fundamental engineering constructions for stabilizing compensator design for rational transfer functions [5,6]. Extension of this theory to irrational transfer functions arising in structural control problems is a focus current research [7,8,9]. Our goal here is to investigate the application of available theory to developing stabilizing controllers for component level models of elastic structures. Such "low authority" controllers can be based on rather simple models for structures where the wave dynamics are of importance. We show that transfer function methods can offer simple solutions to parasitic sensitivities resulting from controller implementation dynamics.

In section 2 we consider certain inherent robustness issues which arise in control of wave models for certain simple linear structural elements. In section 3 we briefly review the transfer function approach to output feedback for rational transfer functions and highlight the extent of development of theory to extend the results to irrational systems. Returning to the wave equation we discuss the implementation of an infinite dimensional controller for the system and show how certain aspects of implementation such as computational delay can be handled in design. Finally, we provide some comments on modeling of elastic structures, controller realization and robustness. This paper is a summary of the report [9] which also includes more detailed examples of control for elastic (Euler-Bernoulli) beams.

2.0 Boundary Control of Wave Models

The small vibration response of a number of simple structural elements (such as cables in tension and rods in compression or torsion) have dynamics which can be modeled by the wave equation with boundary control $u(t)$ and deflection $w(t,x)$ defined on the one dimensional domain $0 \leq x \leq L$ [2];

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$$\begin{aligned} \frac{\partial^2 w(t,x)}{\partial t^2} &= \alpha^2 \frac{\partial^2 w(t,x)}{\partial x^2} \qquad (1)\\ \alpha^2 \left.\frac{\partial w(t,x)}{\partial x}\right|_{x=0} &= 0 \\ \alpha^2 \left.\frac{\partial w(t,x)}{\partial x}\right|_{x=L} &= u(t). \end{aligned}$$

To apply feedback control we take as a measurement the deflection rate at the same end;

$$y(t) = \frac{\partial w(t,L)}{\partial t}. \qquad (2)$$

For example, consider a uniform rod of length L undergoing torsional vibration. Let the torsional stiffness be J_T, mass density ρ, and radius of gyration r. Let the rod be excited by applied torque at one end $u(t) = \tau$. The characteristic parameter (which determines the wave speed) is then $\alpha^2 = \frac{J_T}{\rho r^2}$.

Related studies have been recently completed by von Flotow [10]. In [10] the idea of 'low authority' control for large, flexible space structures is considered based on the characteristic wave propagation for simple structural components. The suggested control strategy is to provide control at the structural junctions between components (i.e. boundary control) which acts to absorb waves thus suppressing standing waves and limiting wave propagation between individual components. The strategy is based on that used in design of high frequency electrical networks and amounts to providing a termination of 'matched impedance'. Experimental results confirm the effectiveness of such simple controllers based on very simple models. Potentially much more sophisticated control laws can also be developed based on such simple models if a clear understanding of the robustness of the closed loop control can be given. Such practical ideas motivate the control problems considered in this paper. Significant issues concerning the robustness of such control problems have recently been raised [4]. Following the examples in [7] we consider these issues from the point of view of transfer function analysis. We also consider the potential for novel control implementations.

Taking Laplace transforms in (1), (2) the transfer function from the boundary control to the deflection rate,

$$\hat{y}(s) = T_o(s)\hat{u}(s), \qquad (3)$$

can be simply derived for this system and put in the form,

$$T_o(s) = \frac{1}{\alpha}\coth\left(\frac{k}{\alpha}s\right) = \frac{1}{\alpha}\,\frac{1+e^{-2\frac{L}{\alpha}s}}{1-e^{-2\frac{L}{\alpha}s}} \qquad (4)$$

with $\alpha > 0$ a real parameter; viz., the wave speed. We have written the irrational transfer function suggestively as a ratio of *exponential polynomials* which we view as a generalization of rational constructions. Clearly such a transfer function has an alternate realization as a delay equation; e.g.,

$$y(t) = y(t-\tau_1) + \frac{u(t)}{\alpha} + \frac{u(t-\tau_1)}{\alpha},$$

with $\tau_1 = 2L/\alpha$. In the sequel we discuss the implication of alternate realizations for control synthesis for some distributed parameter systems of this type. Further discussion of the characterization of wave dynamics as delay systems is given in Laguese [11].

We consider stabilization by rate feedback with the control law

$$\hat{u}(s) = \hat{r}(s) - k\hat{y}(s) \qquad (5)$$

as indicated in Figure 1.

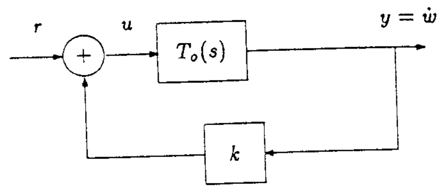

Figure 1: Rate feedback loop for control of wave dynamics

Claim 1 The wave equation (1) with boundary control and colocated observation (2) is exponentially stabilized by the rate feedback control law (5) for any gain $k > 0$.

This result is well known and a proof is given in [11]. The result is easy to see using transfer function analysis. The closed loop transfer function

$$\hat{y}(s) = T_c(s)\hat{r}(s)$$

is given by

$$T_c(s) = \frac{T_o(s)}{1+kT_o(s)} = \frac{\frac{1}{\alpha}\left(1+e^{-2\frac{L}{\alpha}s}\right)}{1-e^{-2\frac{L}{\alpha}s} + \frac{k}{\alpha}\left(1+e^{-2\frac{L}{\alpha}s}\right)} \qquad (6)$$

which is also characterized by delays. To show exponential stability we compute the spectrum of the closed loop system. The singularities of T_c consist of poles which are roots of the characteristic equation

$$1 + \frac{k}{\alpha} + \left(\frac{k}{\alpha} - 1\right)e^{-2\frac{L}{\alpha}s} = 0, \qquad (7)$$

or equivalently roots of

$$e^{-2\frac{L}{\alpha}s} = \frac{1+k/\alpha}{1-k/\alpha} \qquad (8)$$

which has roots as

$$s_n = \frac{\alpha}{2L}\ln\left|\frac{1+k/\alpha}{1-k/\alpha}\right| + i\,\frac{n\pi\alpha}{L}, \qquad (9)$$

for $n = 0, \pm1, \pm2, \ldots$. Since $\alpha > 0$ then for any k such that $0 < k < \infty$ we see that

$$\ln\left|\frac{1+k/\alpha}{1-k/\alpha}\right| < 0$$

and therefore $\Re e\, s_n < 0$ for each n. Equation (9) demonstrates that the choice of feedback gain k permits translation of the spectrum parallel to the real axis in the complex s-plane. In fact, for $k = \alpha$ the spectrum disappears. This condition corresponds to the idea of impedance matching at the termination of elastic elements of structures and is currently under experimental investigation by von Flotow [10].

Recently, Datko et al [4] have demonstrated that the introduction of any small delay in this control loop (due to computational or other sound physical reasons) will destabilize this system for any gain $0 < k < \infty$. Although this result is somewhat delicate to show by direct analysis of the resulting combined partial differential and delay equation it is quite easy to see using transfer function analysis. Indeed, classical analysis of this simple loop shows that the system has an ultimate, *zero phase margin* as $s \to \infty$. The sensitivity can be seen directly from the root pattern of the ideal closed loop system given by (9). The poles, s_n, are periodically spaced along a vertical line so that they have damping ratios $\zeta_n = \frac{\Re e\, s_n}{\Im m\, s_n} \to 0$ as $n \to \infty$.

One way to see the phase sensitivity is to replace the ideal feedback gain k with the delay transfer function $ke^{-s\tau}$ for some $\tau > 0$, with τ real. In the ideal case, $\tau = 0$ and (9) shows that all poles are contained in the open left half plane. To show how instability can result it is enough to show the following:

Claim 2 With k replaced by $ke^{-s\tau}$ in the characteristic equation (7) for any $\alpha > 0$, $k > 0$ there exists a $\tau > 0$ arbitrarily small such that some root of (7) exists on the imaginary axis.

To see this note that (8) with the indicated replacement for k leads to the condition

$$e^{-2\frac{L}{\alpha}s} = \frac{1 + \frac{k}{\alpha}e^{-s\tau}}{1 - \frac{k}{\alpha}e^{-s\tau}}. \qquad (10)$$

To show that this equation has roots for some $s = i\omega$ we first find real frequencies ω_n such that

$$\left| \frac{1 + \frac{k}{\alpha}e^{-i\tau\omega}}{1 - \frac{k}{\alpha}e^{-i\tau\omega}} \right| = 1. \qquad (11)$$

By direct computation we see that (11) is satisfied for

$$\omega_n = (2n+1)\frac{\pi}{2\tau}, \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (12)$$

Substituting $s = i\omega$ with (12) into (10) gives

$$e^{-i(2n+1)\frac{L\pi}{\alpha\tau}} = \frac{1 + i(-1)^{n+1}\frac{k}{\alpha}}{1 - i(-1)^{n+1}\frac{k}{\alpha}}.$$

Comparison of the argument of either side gives

$$(2n+1)\frac{\pi L}{\tau\alpha} = 2(-1)^n \tan^{-1}(k/\alpha).$$

So that the required delay is

$$\tau = \frac{(2n+1)\pi L}{2\alpha(-1)^{n+1}\tan^{-1}(k/\alpha)}$$

and since $0 \le \tan^{-1}(k/\alpha) \le \pi$ for any $k, \alpha > 0$ fixed there is a $\tau > 0$ arbitrarily small which satisfies the above relation for some integer $n = \pm 1, \pm 3, \ldots$.

Phase margin sensitivity of this type is clearly cause for concern if really evident in the physical system. We suggest that the difficulty arises because damping has been ignored in the model. For elastic structures in space applications we expect that material losses will become evident at high frequencies. In the next section we suggest that simple control implementations are possible based on lossless models. We show how concern for such phase sensitivity can be mitigated by judicious use of dynamic compensation.

3.0 Spectrum Reassignment by Output Feedback

One of the most fundamental and constructive methods for control system design for finite-dimensional, lumped parameter systems involves pole relocation by output feedback. It is well known that if the system is both controllable and observable then its poles can be arbitrarily reassigned by a combination of asymptotic state estimation followed by state feedback. Similar constructions can be obtained without explicit identification of a state-space realization (or model) using algebraic methods based on manipulation of rational transfer functions [12].

In [13] Youla et al provide a complete algebraic characterization of *all* stabilizing compensators with rational transfer functions for a given rational plant model. This important result is used to solve optimal control design problems by association with a Wiener-Hopf problem. Youla et al give specific examples demonstrating that the approach based on manipulation of transfer functions (rational matrices in the case of multiloop design) can effectively solve problems where the transfer functions cannot be realized by conventional state space models. Such *improper* transfer functions arise naturally in modeling certain standard control components such as rate gyros. These models are useful for predicting frequency response in limited bandwidths only. Similar effects are evident in standard models for elastic structures.

Let $T_o(s)$ be the system transfer function and assume that

$$T_o(s) = \frac{n_o(s)}{d_o(s)}$$

with n_o, d_o coprime polynomials in s. (This condition corresponds to the assumption of observability and controlability of a state-space realization for T_o.) Consider the control structure as shown in Figure 2;

$$\hat{u}(s) = \hat{r}(s) - H_i(s)\hat{u}(s) - H_o(s)\hat{y}(s), \qquad (13)$$

$$\hat{y}(s) = T_o(s)\hat{u}(s), \qquad (14)$$

which leads to the closed loop transfer function

$$\hat{y}(s) = T_c(s)\hat{r}(s) = \frac{T_o(s)}{1 + H_i(s) + T_o(s)H_o(s)}\hat{r}(s). \qquad (15)$$

The algebraic properties which permit a constructive approach to arbitrary pole reassignment for $T_c(s)$ follows from the fact that the ring $\Re[s]$ [5] is both a principal ideal and an Euclidean domain. By assumption of coprimeness of the factors n_0, d_o we know that a solution exists to the Bezout equation

$$n_o(s)x(s) + d_o(s)w(s) = 1. \qquad (16)$$

Since $\Re[s]$ is an Euclidean domain the Euclidean division algorithm provides a constructive approach to solving the Bezout equation. From (15) we write

$$T_c = \frac{n_o}{d_o + H_i d_o + n_o H_o} \qquad (17)$$

and the pole positioning problem is to find H_i, H_o so that

$$d_o + H_i d_o + n_o H_o = d_c \qquad (18)$$

where d_c is a polynomial with desired roots. With appropriate considerations for polynomial degrees we see that the Diophantine problem (18) can be solved by the choice

$$H_i = w(d_c - d_o) \qquad (19)$$

$$H_o = x(d_c - d_o). \qquad (20)$$

[5] $\Re[s]$ denotes the ring of polynomials in s with real coefficients while $\Re(s)$ is the associated field of rational functions.

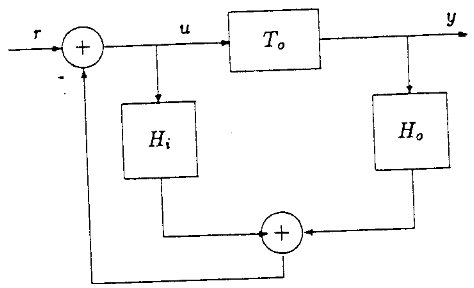

Figure 2: Configuration for output feedback control

With a particular solution in hand for (18), the realization question can be addressed separately. The above analysis is based on algebraic manipulation over the ring of polynomials $\Re[s]$. The required compensator transfer functions involve ideal differentiation (*improper* transfer functions.) Observer-based (asymptotic state estimation followed by state feedback) design replaces these transfer functions with realizable $H_i, H_o \in \Re(s)$ which are proper transfer functions (i.e. bounded at $s = \infty$). In general we can take

$$H_i = \frac{w(d_c - d_o)\varphi}{\varphi}, \quad H_o = \frac{x(d_c - d_o)\varphi}{\varphi}$$

where $\varphi \in \Re[s]$ is chosen so that H_i, H_o are stable and realizable. Realization can be achieved if certain degree requirements can be met on the polynomials $x(d_c - d_o)\varphi$ and $w(d_c - d_o)\varphi$ (cf. Kailath [12, pp. 306–310] for a complete discussion of the algebraic construction.) Roughly put, the degree requirements can be met by reduction of H_i, H_o using Euclidean (polynomial) division when the degree of φ equals the degree of d_o.

In this paper, we demonstrate by example that realization can also be addressed more directly in certain cases. For irrational transfer functions, the algebraic reduction process described above is not always possible. For linear systems with well defined Laplace transforms the algebraic properties can be extended in various ways to form an enlarged *ring* of systems (see [5,14] for examples). Such rings do not typically form an Euclidean domain; i.e., Euclidean division does not "converge" in the sense of degree reduction. However, it may still be possible to realize H_o, H_i appearing in (19), (20) directly using a fast, digital computer. An example is given in the next section.

Explicit solutions for the Bezout equation can be obtained for a more general class of irrational functions; viz., $T_o = n_o/d_o$ when n_o, d_o are entire (i.e., analytic everywhere in the finite complex plane) without Euclidean division. Berenstein and Struppa [15] give an explicit solution to the Bezout equation when the functions $n_o(s), d_o(s) \in \hat{E}(\Re)$, the ring of Laplace transforms of generalized functions (distributions) of compact support in $\Re$. The theory indicates that requirements for coprimeness of polynomials is somewhat more delicate for this class of functions. Similar conditions for *strong coprimeness* are given by Baras[7,8] based on the Carleson Corona theorem. We summarize the theory from the discussion in Berenstein and Struppa [15].

We say the function $p(s)$ satisfies Paley-Wiener estimates if for some $c, a, N > 0$ and for all $s \in C$

$$|p(s)| \le c(1 + |s|)^N e^{a|\mathrm{Im}\ s|}.$$

If n_o, d_o satisfy such estimates then a necessary and sufficient condition for a solution to the Bezout equation (16) to exist is that

$$|n_o(s)| + |d_o(s)| \ge \epsilon(1 + |s|)^{-L} e^{-B|\mathrm{Im}\ s|} \qquad (21)$$

for some $\epsilon, L, B > 0$. This is the case for example if n_o, d_o are *exponential polynomials* of the form $\sum c_k(s)e^{\beta_k s}$ with $c_k(s) \in \Re[s]$, $\beta_k \in C$, the summation is over a finite number of terms, and roots of n_o, d_o are "sufficiently separated". We refer to condition (21) as *strong coprimeness.* The explicit (and rather messy) formulae for a particular solution to the Bezout equation are given in [15].

Algebraic methods for solving the Bezout equation are possible for some irrational functions by introducing multivariate polynomials and solving by elimination theory (cf. Bose [16]). The major problem with such methods is since such functions form a ring which is not necessarily a Euclidean domain that the required degree growth of the polynomial terms at each step can lead to computational explosion which renders the approach impractical for all but the simplest problems.

Spectrum Relocation by Delay Control Realization for Wave Models

Transfer function methods can suggest alternate control realizations which can offer advantages. We return to the boundary control of the wave equation to illustrate the ideas. To the extent that computational delays for realtime control are predictable it is possible to compute dynamic feedback compensation which can provide arbitrary spectrum relocation. By special choice of closed loop spectrum it is possible to realize the controller by a simple digital machine. Constraints on realization are discussed.

Consider the wave model (1) with transfer function $T_o(s)$ given by (4). We consider dynamic feedback compensation of the form (13) (see Figure 3). First, note that by writing $T_o(s)$ in the form (4) we recognize the delay nature of the characteristic response. This suggests a method by which we can explicitly solve the Bezout equation. Moreover, the resulting dynamic controller can be easily realized by a delay system.

Now assume that a computational delay for realtime implementation is anticipated of τ time units. We lump the delay with the wave model and replace the transfer function with the following:

$$T_o(s) = \frac{e^{-s\tau}}{\alpha} \frac{1 + e^{-2\frac{L}{\alpha}s}}{1 - e^{-2\frac{L}{\alpha}s}} = \frac{n_o(s)}{d_o(s)}, \qquad (22)$$

where we identify the coprime factors as,

$$n_o(s) = \frac{e^{-s\tau}}{\alpha}(1 + e^{-2\frac{L}{\alpha}s}) \qquad (23)$$

$$d_o(s) = 1 - e^{-2\frac{L}{\alpha}s}. \qquad (24)$$

For this simple system a particular solution to the Bezout equation (16) can be obtained immediately as

$$x(s) = \frac{\alpha}{2}e^{s\tau}, \quad w(s) = \frac{1}{2}. \qquad (25)$$

The solution of the Bezout equation is in this case facilitated by the understanding that the underlying delay nature of the response in T_o is evident for example by a transformation $e^{2\frac{L}{\alpha}s} \rightarrow z$ which permits solution of the nominal case with $\tau = 0$ by Euclidean division over the ring $\Re[z]$.

Keeping in mind the delay sensitivity problem encountered in constant gain feedback case of Section 2, we consider the problem of achieving a prescribed degree of exponential stability in the presence of the known computational delay. To facilitate simple realization it is then sufficient to seek feedback compensators $H_o(s), H_i(s)$ so that the closed loop characteristic equation has the form,

$$d_c(s) = 1 + e^{-\delta(s+\beta)} \tag{26}$$

which has roots

$$s_n = -\beta - i(2n+1)\frac{\pi}{\delta} \tag{27}$$

for $n = 0, \pm1, \pm2, \ldots$ Obviously more general compensator constructions are possible (in principal) but the resulting class of delay-realizable controllers is of immediate interest both for simplicity of computation and simplicity of implementation.

The desired dynamic compensators can be computed as in (19), (20),

$$\begin{aligned} H_i(s) &= \frac{1}{2}(e^{-\delta(s+\beta)} + e^{-2\frac{L}{\alpha}s}) \\ H_o(s) &= e^{s\tau}\frac{\alpha}{2}(e^{-\delta(s+\beta)} + e^{-2\frac{L}{\alpha}s}). \end{aligned}$$

It is a simple exercise to show that the control law can then be immediately implemented as a digital realization of the difference equation;

$$\begin{aligned} u(t) &= r(t) - \frac{e^{-\delta\beta}}{2}u(t-\delta) - \frac{1}{2}u(t - \frac{2L}{\alpha}) \\ &\quad -\frac{\alpha}{2}e^{-\delta\beta}y(t-(\delta-\tau)) - \frac{\alpha}{2}y(t-(\frac{2L}{\alpha}-\tau)). \end{aligned} \tag{28}$$

This is implementable for $\tau < \min(\frac{2L}{\alpha}, \delta)$.

The parameter τ is a design parameter limited only by the availability of fast, digital computer to implement the control law. The above criterion sets an upper bound on the delay proportional to characteristic length L and inversely proportional to wave speed α.

In applications it may be desired to achieve more general pole patterns than available from (26). However, delay realization of the type illustrated above offers a simple implementation of an effective infinite-dimension controller with a standard digital computer. The idea of delay realization offers a paradigm for controller design which can address additional engineering design considerations such as frequency shaping, optimal quadratic cost and considerations for disturbance rejection and command following. More comprehensive examples are given in [9] including a complete optimal control problem using the algebraic constructions described in [13].

Transformation of the process transfer function (as in (22)) via $e^{2\frac{L}{\alpha}s} \rightarrow z$ permits algebraic computation over a ring of delay realizations ($\Re[z]$) [5]. In general, for such delay realizations the possible pole patterns (roots of $d_c(z)$) will be contained in a vertical strip in the s-plane; i.e., with s_n the roots of $d_c(s) = 0$ then

$$\beta_2 \leq \Re e\ s_n \leq \beta_1$$

for some real $\beta_2, \beta_1 < 0$. For each root of $d_c(z)$ in the z-plane—say z_k—the roots of $d_c(s) = 0$, viz. s_k, are given as roots of

$$e^{2\frac{\alpha}{L}s_k} - z_k = 0$$

or equivalently as

$$s_k = \ln|z_k| + i(\theta_k + 2n\pi) \quad n = 0, \pm1, \pm2, \ldots$$

and $z_k = |z_k|e^{i\theta_k}$. Clearly each root in the z-plane unfolds along vertical lines in the s-plane.

Several remarks are appropriate with respect to such pole patterns occurring naturally in structural element modeling. Using delay realizations of the type described above we see that the fundamental property that the damping ratio approaches zero for arbitrarily high frequency poles will be retained in achievable closed loop transfer functions. Thus the ultimate, phase margin sensitivity is still evident. Such pole patterns are predicted by the theory of hyperbolic PDE's. However, recent experimental work on beams by Russell [17] indicates that these systems are *not* purely hyperbolic. In fact, material losses (which introduce parabolic models) take over at high frequencies. Clearly such models are useful for finite bandwidth analysis only! Our viewpoint is that considering the simplicity of the resulting control law realizations that such models may still be quite useful for control synthesis if we understand the bandwidth limitations (perhaps by physical experiments) of the model or by a more complete understanding of the physics of material damping as pursued in the research of Russell [17].

It is also significant to mention work of Cooke [18] on determining more general realizations for hyperbolic PDE's. In [18] a method is presented for reduction of initial-boundary value problems for hyperbolic PDE's to differential-difference equations. Again, various useful alternate realizations for control synthesis result.

4.0 Perspectives on Modeling and Controller Realization

Classical frequency domain analysis of robustness of feedback focuses on the determination of stability margins. The phenomenon observed in boundary control of the wave equation is clearly a problem of phase margin. Determination of stability margins by classical frequency domain methods based on Nyquist tests is complicated for these models in practice because frequency response data must be available over the entire imaginary axis including the point $s = \infty$ and because no damping is included in the structural wave model. Despite these limitations, the utility of these models for control design and identification of simple controller realization is significant.

Models for Flexible Structures: Bandwidth Considerations

The models discussed in this paper for standard flexible components such as rods and beams yield to some interesting insights using transfer function analysis. The simple models used in this paper each lead to concern over the ability to design robust control for physical applications based on such models. Both models include no inherent damping. More importantly these models have infinite bandwidth in a sense important for analysis of feedback stability and robustness. It is easy to see that the wave equation model with boundary control and observation has infinite bandwidth by rewriting the transfer function (4) in the form

$$T_o(s) = \frac{1}{\alpha}\frac{1+e^{-2\frac{L}{\alpha}s}}{1-e^{-2\frac{L}{\alpha}s}} = -\frac{1}{\alpha} + \frac{2/\alpha}{1-e^{-2\frac{L}{\alpha}s}}.$$

Clearly $\lim_{s\rightarrow\infty} T_o(s) = -\frac{1}{\alpha}$.

One way to describe this property is in terms of the *relative degree* of the transfer function which we define as the number of zeros of $T_o(s)$ at $s = \infty$. The relative degree can be interpreted as having both positive and negative values by considering the Laurent series expansion of $T_o(s)$ about the point $s = \infty$. If $T_o(s)$ has a pole of degree n at $s = \infty$ then the relative degree is taken to be $-n$.

Classical frequency domain analysis such as finite bandwidth Nyquist tests and root locus analysis is based on the assumption that the closed loop pole locations depend continuously on the scalar gain k [19, pp. 49–51]. It is easy to show that for transfer functions with relative degree less than one (such as the example discussed above) that the closed loop poles may vary in a discontinuous fashion with k. In the case considered in Section 2, the return difference

$$1 + kT_o(s) = \frac{d_o(s) + kn_o(s)}{d_o(s)}$$

has relative degree zero for almost all finite k. However, as shown in (9), the case $k = 1/\alpha$ leads to relative degree $-\infty$.

The infinite bandwidth property is evident in transfer function analysis of standard models for structural components (including the Bernoulli-Euler beam) and has been observed by other researchers in the course of analysis of modeling and control of flexible space structures [20]. As pointed out by Wie and Bryson [20] such models typically arise in analysis of time-scaled or singularly perturbed models where certain fast parasitics have been replaced by an instantaneous approximation to their steady state response. Although such models are convenient and can lead to simplified control architectures even for large, but finite dimensional problems, it is well known that analysis appropriate for determination of robustness depends on the nature of ignored parasitics.

One way in which such distributed models can be developed is as the result of asymptotic analysis of large repetitive rigid structures. An example of such an approach is given by Blankenship [21] using the method of "homogenization". The importance of such analysis for design of robust closed loop control is that the parasitics (which involve the local element interactions) can be taken into account in the robustness analysis. Such analyses are currently underway at SEI and will be reported elsewhere.

One obvious source of parasitic dynamics which can affect the bandwidth of the process (i.e. the relative degree) comes from the actuators and sensors. Inclusion of actuator and/or sensor dynamics may lead to a process transfer function with relative degree ≥ 1. A simple example serves to illustrate that such analysis requires some knowledge of the bandwidth of the structural response. Let the actuator/sensor dynamic be modeled as a simple bandwidth limitation so that the process transfer function T_o can be replaced as

$$T_o(s) = \frac{\coth(\frac{k}{\alpha}s)}{\alpha(\tau s + 1)}.$$

The resulting characteristic equation is

$$(1 - e^{-2\frac{L}{\alpha}s})(\tau s + 1) + \frac{k}{\alpha}(1 + e^{-2\frac{L}{\alpha}s}) = 0.$$

We are concerned with the possibility of imaginary roots of this equation. Set $s = i\omega$ in the above and recognize that since $|(1 + e^{-2\frac{L}{\alpha}s})| < 2$ it is easy to see that the roots approach the solutions of

$$(1 - e^{-2\frac{L}{\alpha}i\omega})\tau i\omega = 0$$

as $\omega \to \infty$; i.e., the open loop poles which in this case are undamped. Thus as expected the high frequency poles are unaffected by bandlimited control action. Although this is a realistic assumption we need to include a realistic model for inherent structural damping in order to determine the effective robustness and stability properties of the control system. Models for material damping which lead to bandlimited response of the structural system are not currently available.

Controller Design and Damping Models

It is now clear from the work of Gibson [22] and others that stabilization of distributed systems by approximation of an ideal, infinite dimensional controller for the distributed parameter system is possible only under certain conditions when *compact* boundary feedback is used. In particular, Gibson has shown that the open loop system must have some inherent damping such that the semigroup operator is uniformly exponentially stable. For the models we have considered such damping can be obtained from small "viscous" type damping leading to transfer function being analytic for $\Re e\, s \geq 0$.

The frequency response analysis considered in this paper suggests that for the design of robust control systems for elastic structures that considerations for material damping may also be required. Consider a simple viscoelastic model for a uniform rod under torsional stress. A standard model for material damping results by assuming the stress is a linear function of strain and strain rate, and (1) is therefore replaced by

$$\frac{\partial^2 w(t,x)}{\partial t^2} = \alpha^2 \frac{\partial^2 w(t,x)}{\partial x^2} + \varsigma \frac{\partial^3 w(t,x)}{\partial^2 x \partial t}.$$

The transfer function (4) becomes

$$T_o(s) = \frac{1}{\alpha}\frac{1 + e^{-2\frac{L}{\alpha}\lambda(s)}}{1 - e^{-2\frac{L}{\alpha}\lambda(s)}}$$

where $\lambda^2 = \frac{s^2}{\alpha^2 + \varsigma s}$. A straightforward calculation indicates that $T_o(s)$ has poles p_n and zeros z_n which satisfy

$$p_n^2 + \varsigma(\frac{\pi n}{L})^2 p_n + \alpha^2(\frac{\pi n}{L})^2 = 0$$

$$z_n^2 + \varsigma(\frac{\pi}{L}(\frac{2n+1}{2}))^2 z_n + \alpha^2(\frac{\pi}{L}(\frac{2n+1}{2}))^2 = 0.$$

If follows that for any real $\varsigma > 0$ (no matter how small) and any specified 'damping ratio', ς^* that there exists an integer n^* such that $\frac{\Re e\, p_n}{\Im m\, p_n} > \varsigma^*$ for all $n > n^*$. Using frequency domain analysis it is then easy to show that the delay sensitivity in Claim 2 will not hold and a finite, positive lower bound (depending on ς) on the delay τ necessary to cause instability can be determined.

Controller Realization and Implementation

We have suggested that delay-realization is an approach which appears to offer alternatives for wave dynamics where dispersion and dissipation can be neglected. For certain standard linear beam models (such as Euler-Bernoulli model) wave dynamics include dispersion even when dissipation is ignored. Such parabolic models are not amenable to delay realization. In [9] we provide an example computation of an ideal infinite dimensional compensator for the Euler-Bernoulli beam with boundary control and observation. The ideal controller is computed as the limiting case of a series of truncated product expansions for the meromorphic transfer function models.