QUANTUM FILTERING WITH COUNTING MEASUREMENTS

J. S. Baras
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

ABSTRACT

Continuous time quantum filtering problems are formulated based on ideas from quantum stochastic processes. This formulation extends previous work in that it allows modelling of the interaction between the measurement process and the state of the quantum field. However, the abstract quantum measurements are restricted to those that can be implemented by counting measurements. The type and properties of the resulting point process are discussed. Following ideas of Davies on Quantum stochastic processes the optimal filtering problem, including optimal measurement selection (from the above type) is formulated. Methods of solutions are discussed and examples illustrate the results. The operator differential equation satisfied by the density operator is analyzed and relations with stochastic partial differential equations of a specific type are illustrated.

SUMMARY

In a series of papers we have recently analyzed problems of linear filtering for signals carried by quantum fields [1-5]. The major thrust behind this work is to obtain necessary and sufficient conditions for optimality and performance bounds for the correct model which incorporates quantum models and necessitates nonclassical treatment. In previous [1-2] work the following simplifying assumptions were made to render the problem feasible: (1) The density operator representing the field does not depend explicitly on time, (ii) time is discrete, (iii) the measurement outcomes at different times are independent conditioned on the signal sequence. These simplifications prevent us from incorporating two fundamental issues in the mathematical formulation of the problem: (a) state evolution with time and (b) state-measuremnt interaction. The above assumptions were partially lifted in later work [5], but at the expense of restraining the field states to Gaussian ones and the measurements to canonical ones. A complete quantum treatment of filtering problems requires incorporating (a) and (b) in the mathematical formulation. A general approach to achieve this is presented here, based on the concept of quantum stochastic processes as developed by Davies [6]. A further motivation for such an effort is provided by our desire to understand precisely the implications of simplifying assumptions and formal or "quasi-classical" derivations of multicoincidence statistics of photon-counting experiments [6] on problems of estimation and filtering.

We first discuss models for quantum mechanical measurements ordered by their complexity. Let $\mathcal{H}$ be a complex Hilbert space. Then we denote by $\mathcal{L}(\mathcal{H})$ ($\mathcal{L}_s(\mathcal{H})$) the space of all bounded (and selfadjoint) operators on $\mathcal{H}$; by $\mathcal{T}(\mathcal{H})$ ($\mathcal{T}_s(\mathcal{H})$) the space of all trace class (and selfadjoint) operators on $\mathcal{H}$; by $\mathcal{L}_s^+(\mathcal{H})$, $\mathcal{T}_s^+(\mathcal{H})$ the nonnegative operators in $\mathcal{L}_s(\mathcal{H})$, $\mathcal{T}_s(\mathcal{H})$ [7]. There is a hierarchy (in terms of complexity) of quantum models for measurement. As usual the state of the quantum field is represented by an operator $\rho \in \mathcal{T}_s^+(\mathcal{H})$, with $\mathrm{Tr}[\rho] = 1$. In classical quantum mechanics a measurement is represented by an operator $V \in \mathcal{L}_s(\mathcal{H})$ and the statistics of the measurement outcome are given by the distribution function $F_V(\xi) = \mathrm{Tr}[\rho E_V(-\infty, \xi]]$, where E_V is the spectral measure of V. This formulation (as is by now well known) is inadequate for detection and estimation problems with quantum measurements. Holevo in [8], motivated by approximate simultaneous measurement of non-compatible observables and randomized decision strategies, and independently Davies [9] motivated by repeated measurements on quantum fields, introduced a new formalism. A measurement is represented by a positive operator valued measure $\mathcal{M}$ (p. o. m.) on a measurable space $(U, \mathcal{B})$. A p. o. m. is a mapping $\mathcal{M}: \mathcal{B} \to \mathcal{L}_s^+(\mathcal{H})$ such that for a partition $\{B_i\}$ of U, $\sum_i \mathcal{M}(B_i) = I$. The statistics of the measurement outcome are given by the probability measure $\mu_v(A) = \mathrm{Tr}[\rho \mathcal{M}(A)]$. The physical realization of such a measurement is by adjoining an auxiliary system at a pure state ρ_e and measuring compatible observables on the augmented system; a procedure motivated by Naimark's theorem which asserts that given a p. o. m. $\mathcal{M}$ on $\mathcal{H}$, there exist a pure state ρ_e on a Hilbert space $\mathcal{H}_e$ and a spectral measure E on $\mathcal{H} \otimes \mathcal{H}_e$ such that $\mathrm{Tr}[\rho M(A)] = \mathrm{Tr}[(\rho \otimes \rho_e) E(A)]$, for any ρ. To incorporate the state-measurement interaction, a fundamental issue in filtering problems, a further generalization was introduced by Davies and Lewis [10]. They utilized the mathematical model of a positive map valued measure (p. m. v. m) on a measurable space $(U, \mathcal{B})$, which is a map $\mathcal{E}$ on $\mathcal{B}$, such that $\mathcal{E}(B)$ is a positive linear map of $\mathcal{T}_s(\mathcal{H})$ into itself for every $B \in \mathcal{B}$, $\mathcal{E}(B) \geq \mathcal{E}(\emptyset) = 0$, and for disjoint $B_i \in \mathcal{B}$ $\mathcal{E}(\bigcup_i B_i) = \sum_i \mathcal{E}(B_i)$. A measurement then is represented by an instrument which is a p. m. v. m $\mathcal{E}$ such that $\mathrm{Tr}[\mathcal{E}(U)\rho] = \mathrm{Tr}[\rho]$ for all states ρ. The physical interpretation is clear: $\mathcal{E}(A)\rho$ gives the new state given that the initial state was ρ and that the observed measurement outcome was in the set A, the statistics of the measurement outcome are given by the probability measure $\mu_v(A) = \mathrm{Tr}[\mathcal{E}(A)\rho]$. One can easily establish the fact that for any instrument $\mathcal{E}$ there exist a p. o. m. $\mathcal{M}$ such that $\mathrm{Tr}[\mathcal{E}(A)\rho] = \mathrm{Tr}[\mathcal{M}(A)\rho]$ for all ρ. The p. o. m. $\mathcal{M}$ thus associated to $\mathcal{E}$ is called the <u>measurement performed by the instrument.</u>

Instruments can be composed. What is needed for the filtering problem is a one parameter family of instruments, parametrized by time. This is what we call a <u>quantum measurement process</u> (q. m. p.) [4]. This concept is based on the work of Davies [7] on quantum stochastic processes. There are two cases of interest. In the first, the outcomes form a marked point process [11] and the model is due to Davies [7]. Briefly, let U be the mark space (in the case of photon-counting experiments this will be the position space for the counters), $\mathcal{Y}_t$ be the sample space (i. e. sequences of occurence times and marks (u_i, t_i), $1 \le i \le n$, n free, $0 < t_i < \ldots < t_n \le t$) and $\mathfrak{F}_t$ the usual σ-algebra on $\mathcal{Y}_t$. Let c map $\mathcal{Y}_t \times \mathcal{Y}_s$ onto $\mathcal{Y}_{t+s}$ via concatenation of sample paths. A quantum measurement process with outcomes in $\mathcal{Y}_t$ (quantum stochastic process in Davies's terminology) is a family of instruments $\mathcal{E}_t$ on $\mathcal{Y}_t$ such that (i) $\lim_{t \to 0} \mathcal{E}_t(y_t)\rho = \rho$ for all states ρ and (ii) $\mathcal{E}_s(B)\,\mathcal{E}_t(A)\rho = \mathcal{E}_{t+s}(c(A \times B))\rho$ for $A \in \mathfrak{F}_t$, $B \in \mathfrak{F}_s$. Note that (ii) is the appropriate analog of the Chapman-Kolmogorov equation of classical probability. In the second case the outcomes are allowed to form any classical stochastic process. This slight generalization was introduced in [4] and is similar to the first case except that U is now a complete separable metric space, $\mathcal{B}$ the Borel σ-algebra on U, $\mathcal{Y}_t$ the set of all measurable functions from $[0, t]$ into U and $\mathfrak{F}_t$ the usual σ-algebra on $\mathcal{Y}_t$. The physical interpretation is clear and incorporates the state-measurement interaction. So $\mathcal{E}_t(A)\rho$ is the new state given that the initial state was ρ and that sample path of the outcome process was in $A \subseteq \mathcal{Y}_t$. The one parameter family $T_t = \mathcal{E}_t(y_t)$ forms a semigroup of operators on $\mathcal{T}_s(\mathcal{H})$ describing the evolution of the state as perturbed by measurement. If we let z denote the empty sample path then $S_t = \mathcal{E}_t(\{z\})$ is also a semigroup on $\mathcal{T}_s(\mathcal{H})$ describing the evolution of the state unperturbed from measurements. The basic problem is to characterize the differential version of the effects of measurement. The concept of a quantum measurement process is thus seen to be the most appropriate one for the formulation of the continuous time quantum filtering problem, to which we now turn.

To formulate the general quantum filtering problem, we need to describe in a precise way the "modulation" of the field by the signal. Leaving this particular problem aside for the moment, let x_t be a stochastic process representing the "signal". If we let x^t denote the sample path of x_t up to time t, a quantum field "modulated" by x_t will have a state $\rho(t, x^t)$. We are now given a class of quantum measurement processes $\mathcal{C}_M$ and a class of processing schemes $\mathcal{C}_p$. A processing scheme is a family of functionals f_t of the sample path of the outcome process, denoted $y^t = \{y_s, s \leq t\}$. The general quantum filtering problem is then: given a description of x_t, $\rho(t, x^t)$, $\mathcal{C}_M$, $\mathcal{C}_p$ find a quantum measurement process $\hat{\mathcal{E}}_t$ in $\mathcal{C}_M$ and a processing scheme $\hat{f}_t$ in $\mathcal{C}_p$ so that

$$\hat{x}_t = \hat{f}_t(y^t) \tag{1}$$

is the minimum error variance estimate of x_t.

It is instructive to see how the special assumptions of [1], [2] relate to this formulation. The linear processing schemes of [1], [2] and discrete time translate (1) to

$$f_k(y^t) = \sum_{i=0}^{k} C_i(k)\mathbf{y}(i) \tag{2}$$

so that the nxn matrices $C_i(k)$ characterize f_k. Furthermore the assumption $\rho(t, x^t) = \rho(x_t)$, employed in [1], [2] avoids the difficult problem of modeling the dynamics of state evolution and modulation. This together with the conditional independence assumption, discussed earlier, imply that $\mathcal{C}_M$ consists of quantum measurement processes of the type

$$\mathcal{E}_n(x^n, A)\rho(x(0)) = \left[\prod_{i=0}^{n} \mathrm{Tr} M_i(A_i)\, \rho(x(i)) \right] \rho(x(n)) \tag{3}$$

for $A = A_1 \times A_2 \times \ldots \times A_n$.

That is the quantum measurement process is characterized <u>uniquely</u> by the measurements performed by the instruments at each instant of time. This is as expected since state-measurement interaction is not included in [1], [2]. The problem has been solved in [1], [2] in this special case. Similarly the problem treated in [5] can be better understood using the new general formulation, and this particular case will be presented in this paper also.

We proceed now with the completion of the formulation in the case where the classical outcome process is a marked point process.

The description of "modulation" leads to stochastic operator evolution equations on $\mathcal{T}_s(\mathcal{H})$ as is described in detail in [3], [4]. It is a consequence of the Schrödinger equation that the state of a quantum system evolves according to the evolution equation

$$\frac{\partial \rho(t)}{\partial t} = -i[H(t), \rho(t)] \tag{4}$$

on $\mathcal{T}_s(\mathcal{H})$. We are obviously interested in cases where the Hamiltonian H depends on a stochastic process x_t, the "signal process". To obtain a concrete model and to render the problem feasible we make two assumptions: (i) H depends linearly on x_t, that is

$$H(t, x^t) = \begin{cases} H + x(t)B & \text{(scalar signal)} \\ H + \sum_{i=1}^{n} x_i(t)B_i & \text{(vector signal)} \end{cases} \tag{5}$$

(which is the case for example for electro-optic amplitude or phase modulation of lasers) (ii) x_t satisfies an Itô stochastic differential equation

$$dx_t = a(x_t)dt + b(x_t)dw_t. \tag{6}$$

Then (4) becomes a bilinear stochastic differential equation on $\mathcal{T}_s(\mathcal{H})$ with multiplicative excitation and can be treated satisfactorily in two important cases [3]: (a) when B, B_i in (5) are in $\mathcal{L}_s(\mathcal{H})$ (which includes all fermion quantum fields) (b) when B, B_i are polynomials in the photon creation and annihilation operators (which includes most interesting cases with boson quantum fields). So the evolution of the state of the quantum field, unperturbed by measurements evolves according to

$$\frac{\partial \rho(t)}{\partial t} = -i\,[H(t, x^t), \rho(t)] \tag{7}$$

where $H(t, x^t)$ is given by (5). We want now to combine (7) with the concept of a quantum measurement process in order to derive the complete state evolution including state-measurement interaction. We are inspired again by the work of Davies [7]. He analyzed quantum stochastic processes with bounded interaction rate in the sense that

$$\mathrm{Tr}[\mathcal{E}_t(\mathcal{Y}_t - \{z\})\,\rho] \leq Kt\,\mathrm{Tr}[\rho]\ , \tag{8}$$

and established that the differential relationship between the semigroups T_t, S_t discussed in the previous section is given by

$$Z(\rho) = W(\rho) + J(U)\rho \tag{9}$$

where Z, W are the infinitesimal generators of T_t, S_t and J is a p. m. v. m. The equation

$$\frac{\partial \rho}{\partial t} = \mathbf{Z}(\rho) \tag{10}$$

can be considered as the analog of Kolmogorov's forward differential equation of classical probability theory. We then have extending a result of Davies.

Theorem : Given an Itô stochastic process x_t as in (6) and a quantum field with unperturbed state evolution described by (7), there exists a family of quantum measurement processes parametrized by the sample paths of the signal process x_t such that the state of the quantum field is given by

$$\rho(t) = \mathcal{E}_t(x^t, y_t)\,\rho(o) \tag{11}$$

and satisfies

$$\frac{\partial}{\partial t}\rho(t) = -i[H, \rho(t)] - i\,x(t)[B, \rho(t)] - \frac{1}{2}(R\rho(t) + \rho(t)R) + J(U)\rho(t) \tag{12}$$

in $\mathcal{T}_s(\mathcal{H})$. $R \in \mathcal{L}_s^+(\mathcal{H})$ is the total interaction rate: $\mathrm{Tr}[\rho R] = \mathrm{Tr}[J(U)\rho]$.

This result holds in particular for Boson fields with photon counting measurements. The classical probability measure

$$\mu_t(x^t, B) = \mathrm{Tr}[\mathcal{E}_t(x^t, B)\rho(o)] \tag{13}$$

is the relevant quantity for filtering. In the case of point process outcomes with fermions or bosons and photon counting we then express in this paper the rate of the observed point process

$$\lambda_t(N_t; w_1, \ldots, w_{N_t}; x^t) \tag{14}$$

in terms of $\mathcal{E}_t(x^t, B)$ and then the following program can be executed:

Step 1: Consider state evolution with measurement effects.

Step 2: Construct quantum measurement process model.

Step 3: Compute rate (14) of observed point process.

Step 4: Solve linear or nonlinear classical filtering problem using (14) and (6), (12).

Step 5: Optimize with respect to R and J that characterize the quantum measurement process.

In this paper we also present simple examples of successful application of this methodology which include amplitude and phase modulated lasers, in "classical" and nonclassical states. In addition to the classical coherent states, examples on two-photon coherent states [12] are presented.

Finally the results are related to a quantum derivation of multicoincidence statistics and an investigation is presented on the significance of differences from the classically derived multicoincidence statistics. In certain cases it can be seen that the filtering results with the quantum formulation can be approximated by the classical filtering results in the sense of a perturbation series approximation with higher order correction terms. Implications of the latter on filter performance versus complexity of implementation are briefly indicated.

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