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BILINEAR HEREDITARY DIFFERENTIAL SYSTEMS:
STATE SPACE MODELS IN HILBERT SPACE

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Abstract

We analyze here state space models for bilinear hereditary differential systems of various levels of complexity. It is shown that for many reasons it is more convenient to develop state space models in Hilbert spaces, following the method of Delfour and Mitter. In certain cases explicit bilinear operational differential equations are obtained that illustrate the way the controls influence the spectrum of the infinitesimal generator of the system. Certain connections with the theory of Lie semigroups are also discussed.

Summary

Finite dimensional bilinear systems have been studied intensively during the last few years [1] - [4]. There are many problems where one finds in addition to bilinearity in the dynamics, hereditary behavior. Typical examples are integrated circuits, nuclear reactor dynamics, wave propagation problems in random media [5], [6]. Thus, bilinear hereditary systems represent an important class of nonlinear infinite dimensional systems, and their analysis is feasible. Recently [5], [6] we described certain controllability properties of bilinear systems with constant delays in the state. The formulation in these papers utilizes the space of continuous functions over one delay interval as state space. For the solution of optimal control problems, estimation, filtering and stability analysis it is preferable to represent these systems in state spaces which have the structure of a Hilbert space. This is the purpose of the present paper. Applying the state space theory of general time varying linear hereditary systems of Delfour and Mitter [7] - [9] we construct a state space model for bilinear delay systems in the space M^2:

$$M^2 \triangleq M^2(-b, 0; \mathbb{R}^n) = \mathbb{R}^n \times L^2(-b, 0 : \mathbb{R}^n)$$

with the inner product

$$(h, k) = h^T(0)k(0) + \int_{-b}^{0} h^1(\theta)^T k^1(\theta)\, d\theta$$

Elements in M^2 will be written as $h = (h^0, h^1)$ where h^0 is the $\mathbb{R}^n$ component (i.e. $h(0)$) and h^1 the functional component of h.

We consider first the simple system

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= (A + \sum_{i=1}^{m} u_i(t) B_i)\, x(t) + Cx(t+\tau) \\ x(\theta) &= h(\theta), \quad \tau \le \theta \le 0, \end{aligned}\right\} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, A, B_i, $i = 1, \ldots m$, C, are constant matrices of appropriate dimensions and $u_i(t)$ are scalar functions of time which are measurable and bounded on finite intervals. From [8] it follows that (1) has a unique solution $\varphi(t, s, h, u)$ in the space

$$W^{1,2}_{loc}(0, \infty; \mathbb{R}^n) \triangleq \{f \in L^2(0, T; \mathbb{R}^n) | \; Df \in L^2(0, T; \mathbb{R}^n), \forall T < \infty\} \quad (2)$$

which is linear and continuous with respect to h. Following [8] we let $V \subset M^2$ be the space

$$V \triangleq \{(h(0), h) \in M^2 | \; h \in W^{1,2}(-\tau, 0; \mathbb{R}^n)\}.$$

Then the state of (1) at time $t \ge s$, with initial datum h and controls u_i is defined by

$$\left.\begin{aligned} \tilde{\varphi}(t;s,h,u) &= (\tilde{\varphi}^0(t;s,h,u), \tilde{\varphi}^1(t;s,h,u)) \\ \tilde{\varphi}^0(t;s,h,u) &= \varphi(t;s,h,u) \\ \tilde{\varphi}^1(t;s,h,u)(\theta) &= \begin{cases} \varphi(t+\theta;s,h,u), & t+\theta \ge s \\ h^1(t+\theta-s), & \text{otherwise} \end{cases} \end{aligned}\right\} \quad (3)$$

$\tau \le \theta \le 0$.

The properties of $\tilde{\varphi}(t;s,h,u)$ can be found in detail in [8]. In particular the map $\tilde{\Phi}(t, s;u)$ defined by

$$\tilde{\varphi}(t;s,h,u) = \tilde{\Phi}(t, s;u)\, h$$

is a continuous map $M^2 \to M^2$ and defines an evolution operator [8]. Let now

$\tilde{A}(t) : V \to M^2$

be defined via

$$\left.\begin{array}{l}[\tilde{A}(t)h]^0 = [A + \sum_{i=1}^{m} u_i(t) B_i] h^0 + Ch(\tau) \\ [\tilde{A}(t)h]^1(\theta) = \frac{dh}{d\theta}(\theta), \quad \tau \le \theta \le 0.\end{array}\right\} \quad (4)$$

Then [8], for every h in V the state $\tilde{\varphi}(t;s,h,u)$ is the unique solution in

$$W_{loc}(s,\infty) = \{z \epsilon L^2_{loc}(s,\infty;V) | Dz \epsilon L^2_{loc}(s,\infty;M^2)\}$$

of the state evolution equation

$$\left.\begin{array}{l}\frac{dz(t)}{dt} = \tilde{A}(t) z(t) \ ; \ \text{a.e. in } (s,\infty) \\ z(s) = h .\end{array}\right\} \quad (5)$$

To obtain from (5) an explicit bilinear operational differential equation we introduce the linear operators

$$\left.\begin{array}{l}\tilde{A} : D(\tilde{A}) = V \to M^2, \ [\tilde{A}h]^0 = Ah^0 + Ch(\tau) \\ \qquad [\tilde{A}h]^1(\theta) = \frac{dh(\theta)}{d\theta}, \ \tau \le \theta \le 0.\end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l}\tilde{B}_i, \ i = 1, \dots m; \ M^2 \to M^2, \ [\tilde{B}_i h]^0 = B_i h^0 \\ \qquad [\tilde{B}_i h]^1 = 0.\end{array}\right\} \quad (7)$$

Then we have the following state space model for (1) in M^2

$$\left.\begin{array}{l}\frac{dz(t)}{dt} = (\tilde{A} + \sum_{i=1}^{m} u_i(t) \tilde{B}_i) z(t) \\ z(s) = h\end{array}\right\} \quad (8)$$

This state space model reveals clearly the effects of bilinear control on the evolution of the state of the system. Recall that $\tilde{A}$ as defined in (6) generates a C_0-semigroup [9], and its domain is V. Moreover its structure and the properties of the semigroup it generates are known in detail [10] [11]. On the other hand the operators $\tilde{B}_i$, are compact (actually degenerate [12]) since they have finite dimensional range. In view of the bang-bang theorems of Baras and Hampton [5] [6], we are interested in the evolution of (1) when the controls are piecewise constant. Then we are perturbing the infinitesimal generator of (8) by compact operators, in a way that is clearly controlled by the control functions u_i. Since $\tilde{A}$ has only point spectrum and compact perturbations change eigenvalues only, it is clear that the spectrum of the infinitesimal generator in (8) is controlled by the controls. This reveals clearly the way a system like (1) evolves in state space.

It is easy to see that such an explicit operational differential equation cannot be obtained for (1) if we work in the space of continuous functions as state space. The results above are also valid for various extensions:

1) if we have multiple delays in the state
2) if we have more complicated hereditary behavior i.e. terms of the form

$$\int_{-b}^{0} \Gamma(\theta) \, x(t+\theta) \, d\theta$$

The picture becomes more complicated when we add terms of the form:

1) delays in the controls
2) products of controls and delayed states

Certain results from the theory of Lie semigroups of bounded operators are relevant for the analysis of state space models like those described in this paper. The details and the analysis of more complicated models will be given in [13].

References

1. R. W. Brockett, "System Theory on Group Manifolds and Coset Spaces", SIAM Journal on Control 10, (1972) pp. 265-284.
2. H. J. Sussmann and V. Jurdjevic, "Controllability of Nonlinear Systems", J. of Diff. Eq. 12 (1972) pp. 95-116.
3. R. R. Mohler and A. Ruberti (edts.), Theory and Application of Variable Structure Systems,
4. D. Q. Mayne and R. W. Brockett (edts.), Geometric Methods in System Theory, Reidel Pub. Co., The Netherlands, 1973.
5. J. S. Baras and L. Hampton, "Bilinear Delay Differential Systems", Proceedings of the 1975 Conference on Information Science and Systems, Johns Hopkins Univ., April 1975, pp. 26-32.
6. J. S. Baras and L. Hampton, "Some Controllability Properties of Bilinear Delay - Differential Systems", Proceedings of the 1975 IEEE Decision and Control Conference, Dec. 1975, pp. 360-361.
7. M. C. Delfour and S. K. Mitter, "Hereditary Differential Systems with Constant Delays, I. General case" "II, A class of affine Systems and the Adjoint Problem", J. Differential Eq. 12 (1972) pp. 213-235, J. Differential Eq.
8. A. Bensoussan, M. C. Delfour and S. K. Mitter, Report ESL-P-604, ESL, MIT, June 1975.
9. E. Hille and R.S. Phillips, Functional Analysis and Semigroups, 1957.
10. J. K. Hale, Functional Differential Eqs., 1972.
11. R. B. Vinter, Rept. ESL-R-541, MIT, May 1974.
12. T. Kato, Perturbation Theory for Linear Operators, Springer Verlag, New York, 1966.
13. J. S. Baras, "Bilinear Hereditary Differential Systems: State Space Models in Hilbert Space", in preparation.