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NATURAL MODELS FOR INFINITE DIMENSIONAL SYSTEMS AND THEIR STRUCTURAL PROPERTIES

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Summary

We study here multi-input, multi-output distributed parameter systems. The state space has the structure of a Hilbert space, and the evolution operators form a strongly continuous semigroup. We are thus able to include a large class of systems governed by linear partial differential equations. We present realizability conditions for the input-output maps considered and investigate canonical realizations and their properties. The main mathematical tools are invariant subspace theory in vectorial Hardy Spaces. Using the methods of this particular branch of operator theory we are able to classify the transfer functions considered according to their singularities. This classification is also related to system theoretic concepts and especially to the existence of spectrally minimal realizations. Finally we discuss the implications of these results to the structure theory of distributed parameter systems, to lumped-distributed network synthesis and delay systems.

1. Balanced, Regular and Canonical Realizations

This paper is a continuation of our previous work on infinite dimensional realization theory [1-3]. Operator theory was always closely related to system and network theory. The interaction of these three disciplines has attracted through the years the attention of many researcher's, mathematicians as well as electrical engineers. For the mathematician these two applied fields represent a relevant topic to further stimulate research in operator theory. On the other hand electrical engineers use operator theory to formulate and solve many problems in the analysis and synthesis of networks and systems in general. An example of the latter are the recent developments in the realization theory of distributed parameter systems [1-7] using operator theoretic tools from [8-9].

Here we study multi-input, multi-output distributed parameter systems with state spaces that admit the structure of a Hilbert space. In addition the systems under consideration are linear and time invariant. So we have the description

$$\left.\begin{aligned} \frac{d}{dt} x(t) &= A\, x(t) + B\, u(t) \\ y(t) &= C\, x(t) \end{aligned}\right\} \qquad (1)$$

where: $x(t) \in X$ (a Hilbert space), A generates a C_0-semigroup of bounded operators on X [10], $u(\sigma) \in U$ a finite dimensional Hilbert space (which we will identify with $\mathbb{C}^m$) and $y(t) \in Y$ a finite dimensional Hilbert space (which we will identify with $\mathbb{C}^n$). The input functions are square integrable U-valued functions with compact support. The input-output behaviour of system(1) is given by

$$y(t) = \int_0^t T(t-\sigma)\, u(\sigma)\, d\sigma \qquad (2)$$

where $T(\cdot)$ is a matrix valued function, usually called the weighting pattern, while its Laplace transform $\hat{T}$ is the transfer function. We denote by $\mathcal{L}(X_1, X_2)$ the space of all continuous linear operators mapping the Hilbert space X_1 into the Hilbert space X_2.

Whenever (1), (2) describe the same system we will say as usual that $\{A, B, C\}$ is a realization of T. That is whenever $T(t) = Ce^{At}B$, where e^{At} is a common notation for the C_0-semigroup generated by A. We single out the following two important cases [1]: a realization $\{A, B, C\}$ is regular whenever $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$; a realization $\{A, B, C\}$ is balanced whenever $B \in \mathcal{L}(U, X)$ and R(B) (the range of B) is included in the domain of A (denoted by D(A)), while C is linear and A-bounded [10] (i.e. $D(A) \subseteq D(C)$ and $\|Cx\|_Y \le k_1 \|Ax\|_X + k_2 \|x\|_X$, for some positive constants k_1, k_2 and $x \in D(A)$). Many examples of regular realizations can be found in the book by Lions [11]. They usually come from distributed parameter systems with distributed control and observation. Examples of balanced realizations can readily constructed by considering various partial differential equations and relatively bounded operators (see [10, 12]). They include many interesting cases of boundary observations with distributed control.

Although balanced and regular realizations can describe quite different physical situations, the classes of input-output maps they characterize coincide, as the following theorem indicates.

Theorem 1: Let T be a matrix valued function with values in $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$. Then T has a balanced realization if and only if it has a regular realization. Moreover the infinitesimal generators in both realizations can be taken to be the same.

Proof: This is a generalization of Theorem 3 in [1]. Indeed if $\{A, B, C\}$ is a regular realization of T, then with λ in $\rho(A)$ (the resolvent set of A) we have the balanced realization $\{F, G, H\}$ of T where

$$F = A, \quad G = (\lambda I - A)^{-1} B, \quad H = C(\lambda I - A) \qquad (6)$$

Conversely let $\{F, G, H\}$ be a balanced realization of T, then we get the regular realization $\{A, B, C\}$ where by

$$A = F, \quad B = (\lambda I - F) G, \quad C = H(\lambda I - F)^{-1} \qquad (7)$$

and $\lambda \in \rho(A)$. Clearly B is closed and everywhere defined and therefore bounded by the closed graph theorem. Moreover using the fact that H is A-bounded, it is easily shown that C is closed and since it is everywhere defined, it is bounded.

Remark 1: We did not use in the proof the finite dimensionality of the input and output spaces, and therefore this theorem is true for U, Y infinite dimensional as well.

Definition: The balanced (resp. regular) realization constructed from the regular (balanced) realization by (6) (by (7)) will be called the associated balanced (regular) realization to the given regular (balanced) realization.

In the effort to choose simplified models one defines as usual a realization $\{A, B, C\}$ to be controllable if $B^* e^{A^* t}x=0$ for $t \geq 0$, implies $x=0$, and observable if $Ce^{At}x = 0$ for $t \geq 0$, implies $x = 0$. A realization is canonical whenever it is controllable and observable. Given a realization (regular or balanced) it is an easy matter to obtain a canonical one. This is described in the following theorem which we give without a proof since it is a straightforward generalization of previous results [3, 13].

Theorem 2: Let $\{A, B, C\}$ be a regular realization of a matrix weighting pattern T with state space X. Let M be the orthogonal complement in X of the subspace $M_1 = \{x \in X,\ Ce^{At}x = 0 \text{ for } t \geq 0\}$ and P_M the associated orthogonal projection. Let N be the orthogonal complement in M of the subspace $N_1 = \{x \in M, B^* e^{A^* t}x = 0 \text{ for } t \geq 0\}$ and let P_N be the associated orthogonal projection. Then $\{P_N A|_N, P_N B, CP_N\}$ is a canonical regular realization for T, with state space N.

In the statement of the theorem $P_N A|_N$ denotes the restriction of A on N. It is well defined since it is easily shown that D(A) and N have a dense (in N) intersection.

It can be shown that the associated balanced (regular) realization to a regular (balanced) realization is canonical whenever the regular realization is. This theorem therefore provides also a reduction for balanced realizations.

2. Vectorial Hardy Spaces and Realizability Criteria

In this section we characterize the matrix valued weighting patterns that admit realizations like those described in the previous section. In addition we give some background material on the so called vectorial Hardy Spaces.

The following theorem provides a preliminary characterization, and shows explicitly the limitations imposed on the weighting pattern when it admits such realizations.

Theorem 3: Let T be a matrix valued function, with values in $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$. Then if T is realizable, it is continuous and of exponential order (i.e. each element of the matrix is like that). A sufficient condition for realizability is that every element of T be locally absolutely continuous and that the derivative of T be of exponential order.

Proof: This is a straightforward generalization of Theorem 4 in [1] and we omit the proof.

To proceed we need some background and notation on vectorial Hardy spaces (for more details see [8, 14, 15]. If $\mathcal{K}$ is a Hilbert space (usually separable), then $L_2(\mathcal{K})$ denotes the space of all weakly measurable $\mathcal{K}$-valued functions, with square integrable $\mathcal{K}$-norms. That is $L^2(\mathcal{K}) = L^2((0, \infty); \mathcal{K})$. The Fourier transform of $L^2(\mathcal{K})$ we denote by $H^2(\mathbb{I}; \mathcal{K})$, where $\mathbb{I}$ is the imaginary axis. Every element f in $H^2(\mathbb{I}; \mathcal{K})$ has an analytic extension in the right half-plane Π^+ and

$$\sup_{\sigma > 0} \int_{-\infty}^{\infty} \|f(\sigma + i\omega)\|^2_{\mathcal{K}} d\omega \leq M < \infty$$

These analytic extensions form the space $H^2(\Pi^+; \mathcal{K})$, every element of which has strong limits a.e. on the imaginary axis as $\mathrm{Re}\, s \to 0$. As usual we will often refer to $H^2(\mathcal{K})$, and it will be clear from the context whether we refer to the space of analytic functions or to the space of boundary values. Similarly $H^\infty(\mathcal{K})$ denotes the space of all bounded $\mathcal{K}$-valued analytic functions in Π^+ or their boundary values on the imaginary axis. Also $K^2(\mathcal{K})$ and $K^\infty(\mathcal{K})$ denote the spaces of $\mathcal{K}$-valued functions analytic in the left half-plane Π^- which satisfy similar norm conditions as the functions in $H^2(\mathcal{K})$ and $H^\infty(\mathcal{K})$. Functions in $H^2(\mathcal{K})$, $H^\infty(\mathcal{K})$ are usually called analytic, while functions in $K^2(\mathcal{K})$, $K^\infty(\mathcal{K})$ are called co-analytic. By $H^2(\mathcal{L}(X_1, X_2))$ we understand the space of weakly measurable (in the operator sense) $\mathcal{L}(X_1, X_2)$ valued functions (on the imaginary axis) which are square integrable on the imaginary axis and have analytic extensions in Π^+. Similarly $H^\infty(\mathcal{L}(X_1, X_2))$ is defined. The left translation semigroup on $L^2(\mathcal{K})$ is unitarily equivalent (via Fourier transforms) to the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathcal{K})$' on $H^2(\mathcal{K})$; and similarly the right translation semigroup on $L^2(\mathcal{K})$ to the semigroup 'multiplication by $e^{-i\omega t}$' on $H^2(\mathcal{K})$. We denote by $P_{H^2(\mathcal{K})}$ the orthogonal projection from $L^2(\mathbb{I}; \mathcal{K}, \frac{d\omega}{2\pi})$ onto $H^2(\mathcal{K})$.

In view of the necessary conditions described in Theorem 3, we restrict for the rest of the paper to transfer functions that belong to $H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)) \cap H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. This does not restrict the generality of the discussion (all that is envolved is an appropriate exponential factor or equivalently shifting of the imaginary axis).

Theorem 4: Let T be a matrix valued function which is continuous and such that $\hat{T}$ belongs to $H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)) \cap H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. A sufficient condition for T to be realizable is that $\hat{T}$ has a factorization $\hat{T}(i\omega) = C(i\omega)^* B(i\omega)$ a.e. on the imaginary axis, where $C \in H^2(\mathcal{L}(\mathbb{C}^n, N))$ and $B \in H^2(\mathcal{L}(\mathbb{C}^m, N))$ and N is an auxiliary Hilbert space.

Proof: We take as state space X the Hilbert space $H^2(N)$, and F, G, H as shown below

$$(Gu)(i\omega) = B(i\omega)u$$
$$e^{Ft}x = P_{H^2(N)} e^{i\omega t}x \tag{8}$$
$$Hx = \frac{1}{2\pi}\int_{-\infty}^{\infty} C^*(i\omega)x(i\omega)d\omega$$

Clearly $G \in \mathcal{L}(\mathbb{C}^m, X)$, $H \in \mathcal{L}(X, \mathbb{C}^n)$ and e^{Ft} is a C_0-semigroup. To complete the proof observe that

$$T(t)u = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{T}(i\omega)u d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} C^*(i\omega)e^{i\omega t}B(i\omega)u d\omega$$

for all $u \in \mathbb{C}^m$.

We will frequently refer to the realization described by (8) as the translation realization. The following discussion has two aims. Firstly we want to indicate under what additional assumptions the condition of Theorem 4 becomes necessary. Secondly we would like to investigate relations between the above factorization condition and properties of the models of the system.

Let us first of all observe that Theorem 4 is a generalization of the conditions described in Theorem 3. Indeed the sufficient conditions of Theorem 3 imply that after multiplication by a proper exponential factor, T and $\frac{d}{dt}T$ are in $L^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. If in addition $T(0) = 0$ we have the factorization

$$\hat{T}(i\omega) = \left((1+i\omega)^{-1} I_{\mathbb{C}^n}\right)^* (1-i\omega)\hat{T}(i\omega) \tag{9}$$

and here $N = \mathbb{C}^n$. In case $T(0) \neq 0$, the problem can be reduced to the previous one and solved using (9) for an auxiliary weighting pattern.

Suppose we know appriori that T comes from a system like (1) which is dissipative, in the sense that the operator A is dissipative, that is

$$\langle Ax, x\rangle + \langle x, Ax\rangle \leq 0 \text{ for } x\in D(A) \qquad (10)$$

(the inner product being that of X), and globally asymptotically stable, in the sense that $\lim_{t\to\infty}||e^{At}x|| = 0$ for $x\in X$. In this case following [15] consider the new norm

$$||x||_N^2 = -\langle Ax, x\rangle - \langle x, Ax\rangle \text{ for } x\in D(A)$$

then for $x\in D(A)$, $e^{At}x\in D(A)$ for $t \geq 0$ and $||e^{At}||_N^2 = -2\,\mathrm{Re}\langle e^{At}x, Ae^{At}x\rangle =$

$$= -\frac{d}{dt}||e^{At}x||_X^2$$

and so $\int_0^\infty ||e^{At}x||_N^2 = ||x||_X^2$.

That is if we let N denote the completion of D(A) under the new norm then the map

$$\begin{aligned} P:\ & X \to L^2(N) \\ & x \to h(\sigma) = e^{A\sigma}x \end{aligned} \qquad (11)$$

is an isometry and since D(A) is dense in X can be extended to the whole space X. Moreover

$$Pe^{At}x = e^{Ft}Px \qquad (12)$$

where e^{Ft} is the left translation semigroup on $L^2(N)$. Since P is an isometry its range R(P) is closed, in fact it is a left translation invariant subspace of $L^2(N)$ which we denote by X_1. So P as a map from X to X_1 = R(P) has abounded inverse. Therefore we obtain a realization with state space X_1 and e^{Ft}= left translation, G = PB and H = CP^{-1}. Applying Fourier transforms $\mathfrak{F}$, we get a realization on $X_2 = \mathfrak{F}X_1$ (a subspace of $H^2(N)$) with $H_1 = CP^{-1}\mathfrak{F}^{-1}$ and e^{F_1t} = multiplication by $e^{i\omega t}$ and $G_1 = \mathfrak{F}PB$. Then there exists $B_1\in H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ such that $(G_1u)(i\omega) = B_1(i\omega)u$ for all $u\in\mathbb{C}^m$. Similarly there exists $C_1\in H^2(\mathcal{L}(\mathbb{C}^n, N))$ such that $H_1x = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_1^*(i\omega)x(i\omega)d\omega$ for all $x\in H^2(N)$. Therefore

$$\hat{T}(i\omega) = C_1^*(i\omega)B_1(i\omega) \qquad (13)$$

For reasons that will become obvious in the sequel we are interested in realizations {A, B, C} which are canonical and moreover there exists an a>0 such that the integral $\int_0^\infty e^{A^*t}C^*Ce^{At}e^{-at}dt$ exists and defines a bounded operator on X, denoted by M_{AC}, which is bounded from below. These are controllable and exactly observable realizations (see Helton (7) and also Balakrishnan [13]). The last requirement expresses the property that the initial state be determined by knowledge of the input and the output in a stable way. Their importance lies in the fact that any two controllable and exactly observable realizations of the same weighting pattern T, say {A, B, C} on X_1 and {F, G, H} on X_2 are related via $PAP^{-1}=F$, $PB=G$, $HP=C$ with a boundedly invertible, bounded operator P from X_1 to X_2, [7, 17]. (The State Space Isomorphism Theorem).

Suppose that a matrix weighting pattern (in the class we are studying) has a controllable and exactly observable realization. Then following [13, p. 113] we construct the following realization using left translations. The state space is the closure of the image of $L_2(\mathbb{C}^m)$ functions with compact support under the Hankel operator

$$(H_Tu)(t) = \int_0^\infty T(t+\sigma)u(\sigma)d\sigma \qquad (14)$$

which is well defined and bounded under our assumptions. That is $X=\overline{R(H_T)}$ and is a left translation invariant subspace of $L^2(\mathbb{C}^n)$.

$$\begin{aligned} & e^{Ft} = \text{left translation semigroup on } X \\ & (Gu)(\sigma) = T(\sigma)u, \quad Hx = x(0) \end{aligned} \qquad (15)$$

It is easily shown that the existence of a controllable and exactly observable realization for T is equivalent to the operator 'evaluation at 0' being bounded on $\overline{R(H_T)}$ [13]. Moreover by the state space isomorphism theory all other realizations of T, of this type, differ from (15) by a similarity.

Applying Fourier transforms on (15) we get the following realization in $H^2(\mathbb{C}^n)$

$$X=\overline{R(H_{\hat{T}})},\ e^{At}=e^{i\omega t}|_X,\ (Bu)(i\omega)=\hat{T}(i\omega)u,\ Cx=\frac{1}{2\pi}\int_{-\infty}^{\infty}x(i\omega)d\omega \qquad (16)$$

where $H_{\hat{T}}: H^2(\mathbb{C}^m)\to H^2(\mathbb{C}^n)$ is the Hankel operator associated with $\hat{T}\in H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$

$$H_{\hat{T}}u = P_{H^2(\mathbb{C}^n)}\hat{T}(\mathfrak{I}u) \text{ and } (\mathfrak{I}u)(i\omega) = u(-i\omega).$$

The realization described in (15), (16) will be called the restricted translation realization. From (16) we see by a similar argument as before that we have a factorization $\hat{T}(i\omega)=F_1(i\omega)^*F_2(i\omega)$ as described in Theorem 4. We have thus the following.

Theorem 5: Let $\hat{T}$ be a matrix valued transfer function which belongs to $H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))\cap H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$.

If either a) $\hat{T}$ has a dissipative and stable realization or b) $\hat{T}$ has a controllable and exactly observable realization then $\hat{T}(i\omega)=C^*(i\omega)B(i\omega)$ a.e. where $B\in H^2(\mathcal{L}(\mathbb{C}^m, N))$ and $C\in H^2(\mathcal{L}(\mathbb{C}^n, N))$ and N is an auxiliary Hilbert space.

In Theorems 4, 5, N is a Hilbert space which may very well be of infinite dimension. If N can be taken finite dimensional then it follows that we can realize T by placing together ℓ copies of the one dimensional translation semigroup where ℓ is the dimension of N.

Definition: The minimum dimension for N for which a factorization like the one appearing in Theorem 4 exists, will be called the multiplicity of the weighting pattern T.

There is a large literature for factorizations of operator valued functions in $H^\infty(\mathcal{L}(H_1, H_2))$ [9, 16].

It appears that the multiplicity of a weighting pattern is related to the smoothness of T. This is based on the observation that when T satisfies the smoothness conditions of Theorem 3, then the multiplicity is finite. For the rest of this paper we restrict to weighting patterns of finite multiplicity.

3. Spectral Analysis

The analysis of the structure of a linear distributed parameter system like (1), (i.e. decomposition into subsystems, instabilities etc.) depends greatly on the available information about the spectrum of the infinitesimal generator A. On the other hand from the input-output point of view this information should be directly related to the analytic properties of $\hat{T}$. Thus the need for models with spectral properties that reflect the analytic properties of $\hat{T}$ is evident. The use of such models for the contruction of approximate models is self-evident. When the system displays some internal symmetry, for example A is self adjoint, a very satisfactory theory can be developed (see [1] for details). In the general case however the situation is much more complicated.

Whenever $\{A, B, C\}$ is a realization of $\hat{T}$, we have $\hat{T}(s) = C(Is-A)^{-1}B$ for $Re s > \omega$, for some $\omega > 0$. Then if we let $\sigma(\hat{T})$ denote the set of points of nonanalyticity of $\hat{T}$ and $\rho_0(A)$ the principal connected component of the resolvent set of A we have the spectral inclusion property [1], $\sigma(\hat{T}) \subseteq \sigma_0(A)$ where $\sigma_0(A)$ denotes the complement of $\rho_0(A)$. A realization $\{A, B, C\}$ is called spectrally minimal (c.f. [1]) if $\sigma(\hat{T}) = \sigma(A)$ for some analytic continuation of $\hat{T}$ in the left half-plane. In infinite dimensional systems we encounter often transfer functions that have many possible continuations due to branch points. We want to investigate here the existence of spectrally minimal realizations for the class of transfer functions we study. In the scalar case [3] every noncyclic transfer function has spectrally minimal realizations. The property of noncyclicity is equivalent to the existence of a meromorphic pseudo continuation of bounded type in Π^- [18]. (Recall that the transfer functions we are studying are analytic in Π^+).

We need some background on vectorial Hardy spaces on half-planes [8, 1]. A function G in $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ is outer if the range of the operator M_G, where $M_G f = Gf$ for $f \in H^2(\mathbb{C}^m)$ is dense in $H^2(\mathbb{C}^n)$. A function U in $H^\infty(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n))$ is inner if $U(i\omega)$ is a unitary operator for almost all ω. Every function F in $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ has a factorization $F = U.G$ where U is **rigid** and G is outer and the factors are unique up to a constant unitary factor from the right for U and from the left for G. We have a decomposition of inner-functions to a Blaschke product part and singular part, but the situation here is complicated due to the non-commutativity. For detailed description we refer to Potapov [19]. For a matricial inner function U, det U is a scalar inner function which determines the structure of U to a great extent (see Helson [8] p. 80), and has the property that $(\det U)H^2(\mathbb{C}^n) \subset UH^2(\mathbb{C}^n)$. By a theorem of Lax [14] any right translation invariant subspace M of $H^2(\mathbb{C}^n)$ (i.e. invariant under any $H^\infty(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n))$ function) is of the form $QH^2(\mathbb{C}^n)$ for some matrix valued function Q, which is analytic in Π^+, **bded by 1, and partially isometric on $\mathbb{I}$, with fixed initial space. These** are called rigid. Q is unique modulo a unitary factor from the right. If the invariant subspace is of full range, that is for almost all ω the span of $f(i\omega)$, $f \in M$ equals $\mathbb{C}^n$, then Q is actually inner. Two functions F_1, F_2 in $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ are left (right) prime if they do not have a common nontrivial inner factor from the left (right) in $H^\infty(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ $(H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)))$.

In view of the state space isomorphism theorem for transfer functions that admit controllable and exactly observable realizations, the structure of any such realization is determined once we know the structure of the restricted translation realization. Moreover even for transfer functions that satisfy our general realizability conditions (Theorem 4) the structure of the translation realization can provide useful information about the system. Although we could treat the general case here we prefer due to space limitations to give the details for the class of transfer functions which admit controllable and exactly observable realizations. Since $\overline{R(H_{\hat{T}})}$ is a left translation invariant subspace it is of the form $QH^2(\mathbb{C}^n)^\perp$ for some rigid function Q, which is uniquely defined (modulo a unitary factor). We are mainly concerned with transfer functions $\hat{T}$ that have the property that Q is inner. Fuhrmann in [20, 21] has shown that this is the right generalization of the notion of noncyclicity of the scalar case. Such functions are called strictly noncyclic. From the network theory point of view the importance of these functions is also demonstrated in [22, 23]. A function F is meromorphic of bounded type in Π^- if it has the form $F = \frac{G}{g}$ where G is in $K^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ and g in K^∞ (i.e. scalar). A function E in $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ has a meromorphic pseudo continuation of bounded type if there exists F which is meromorphic of bounded type in Π^- such that $\lim_{Re s \to 0^-} F(s)u = \lim_{Re s \to 0^+} E(s)u$ for all u in $\mathbb{C}^m$. As a direct consequence of Theorem 3.1 in [21] we have

Theorem 6: Let $\hat{T}$ be in $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. Then the following are equivalent:

a) $\hat{T}$ is strictly noncyclic

b) $\hat{T}$ has a meromorphic pseudo continuation of bounded type in Π^-.

c) $\hat{T}$ has a factorization $\hat{T}(i\omega) = U(i\omega)H^*(i\omega)$ with U inner in $H^\infty(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n))$ and H in $H^\infty(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m))$ and such that U and H are right prime.

Moreover if $\hat{T}$ is also in $H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ then H is in $H^2(\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m))$ also. U is uniquely determined modulo a unitary factor from the right and $\overline{R(H_{\hat{T}})} = (U H^2(\mathbb{C}^n))^\perp$ (where $H_{\hat{T}}$ is the Hankel operator associated with $\hat{T}$, see equation (16)). As in the scalar case U is called the associated inner function of the strictly noncyclic function $\hat{T}$.

By a generalization of a theorem of Moeller (c.f. [14] p. 69) we know that the spectrum of the infinitesimal generator of the semigroup 'multiplication by $e^{i\omega t}$' restricted on $(UH^2(\mathbb{C}^n))^\perp$ consists of

a) the points μ in Π^- where $U^*(-\bar{\mu})$ has non null kernel.

b) the points on the imaginary axis through which U cannot be continued analytically to Π^-.

But by Th. 6, $\hat{T}(i\omega) = U(i\omega)H(i\omega)^*$ and the right hand side has the meromorphic continuation in Π^- given by $U^*(-\bar{s})^{-1}H^*(-\bar{s})$, which describes completely the singularities of $\hat{T}$ in Π^-. Using the Potapov expressions for U, and due to the right primeness of U and H it is shown (we omit the details from here) that the singularities of $\hat{T}$ in Π^- are given by the points μ, where $U^*(-\bar{\mu})$ has no inverse. Also by a generalization of an argument in [14] p. 72 we see that $\hat{T}$ has an analytic continuation in Π^- through $i\omega$ if and only if U does. So we have the following.

Theorem 7: Let $\hat{T}$ be in $H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)) \cap$ $\cap H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ and strictly noncyclic. Then the restricted translation realization for $\hat{T}$ is spectrally minimal.

4. Conclusions and Related Problems

We considered here multivariable distributed parameter systems in Hilbert space. With the aid of invariant subspace theory in vectorial Hardy spaces we developed realizability criteria, studied canonical models and investigated the existence of spectrally minimal realizations for the class of transfer functions considered here.

Let us observe that the restricted translation realization (equation (16)) can be defined for any transfer function $\hat{T}$ which belongs to $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. That is we can drop the requirement that $\hat{T} \in H^2(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ also. The resulting realization is neither regular nor balanced. The operators B, C are unbounded.

On the other hand the theory can be developed, in that case, using the rich structure of $H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ as a ring or as a Banach algebra, in a much more algebraic fashion. This can be generalized even further by introducing the following algebra $A^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. Elements of this algebra are functions which belong to $H^\infty(\Pi_\rho^+; \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ for some $\rho > 0$, where Π_ρ^+ denotes the half plane $\operatorname{Re} s > \rho$. Functions that are analytic continuations of one another are identified. Clearly this algebra includes all transfer functions of importance. Much of the structure of H^∞ can be transferred to A^∞. There are advantages in an algebraic approach, but the major gain will be the inclusion in the theory of certain very common transfer functions, which are not included in the present status of the theory. We have in mind delay systems. It is hoped that in this way a complete realization theory for delay systems can be developed.

The connections between infinite dimensional realization theory and lumped-distributed network synthesis become very strong, when invariant subspace theory is used as a common framework (see [24]). DeWilde [22] has developed precise synthesis results for distributed networks using these tools. The problem is far from solved however. A detailed analysis of networks arising in practice is needed, in order to determine the physically meaningful conditions for the mathematical theory. We note the role played by strictly noncyclic functions here (roomy in DeWilde's terminology). Sz-Nagy-Foias [9, 25] have developed Jordan models for certain classes of bounded operators on Hilbert space, similar to the Jordan canonical form of matrix theory. These results can be lifted for the corresponding class of semigroup generators via the Cayley transform. The restricted translation realization in particular will be amenable to such a study. The structure of the system will be determined by the structure of the associated inner function. In this way an invariant factor analysis and a structure theory, parallel to the finite dimensional one can be developed.

Finally it would be of interest to investigate the implications of the available infinite dimensional realization theory to the construction of approximate models and infinite dimensional linear filtering.

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