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Reachability Analysis for Linear Discrete Time Set–Dynamics Driven by Random Convex Compact Sets

Saša V. Raković, Ion Matei and John S. Baras

Abstract— **This paper studies linear set–dynamics driven by random convex compact sets (RCCSs), and derives the set–dynamics of the expectations of the associated reach sets as well as the dynamics of the corresponding covariance functions. It is established that the expectations of the reach sets evolve according to deterministic linear set–dynamics while the associated dynamics of covariance functions evolves on the Banach space of continuous functions on the dual unit ball. The general framework is specialized to the case of Gaussian RCCSs, and it is shown that the Gaussian structure of random sets is preserved under linear set–dynamics of random sets.**

I. Introduction

Modern control theory has recognized the importance and relevance of reachability analysis. The relevance and importance of reachability analysis stem from its intimate relationship with optimal control, set–membership state estimation, safety verification and control synthesis under uncertainty. Indeed, the analysis of uncertain constrained dynamics based on the concepts of reachability enables one to guarantee *a-priori* relevant robustness properties such as robust constraint satisfaction, robust stability and convergence and recursive robust feasibility. The main research topics in reachability analysis include both the characterization and computation of the exact and approximate reachable sets and tubes [1]–[3]. A more detailed exposition to reachability concepts can be found, for instance, in [1], [2], [4]–[7].

Reachability analysis is traditionally performed within the deterministic set–membership setting [2], [4], [5]. The corresponding notions and results are valid as long as the involved constraint sets are known exactly (implying, consequently, that they should be such that the physical aspects of the considered problem are captured in an absolute sense). However, in a variety of important applications including robust design and manufacturing, collaborative robust control and image understanding, the perfect knowledge of the involved sets is seldom available. This paper aims to "stochastify" the inherently deterministic reachability notions. Our approach is based on the well–developed theory of random sets [8]–[15] and on the recent set–dynamics framework for the set invariance under output feedback [16] and for the theory of the minimal invariant sets [6]. In this sense, we study the effect of the randomization of the sets of possible initial states and the disturbance constraint sets within the setting of autonomous linear discrete time dynamics driven by additive disturbances; this is done by considering the associated linear set–dynamics driven by RCCSs. In particular, we study the behaviour of random sets generated by linear set–dynamics, and we derive the set–dynamics of the associated expected reach sets as well as the dynamics of the corresponding covariance functions. More specifically, we show that the expected reach sets evolve according to deterministic linear set–dynamics while the associated dynamics of covariance functions evolves on the Banach space of continuous functions on the dual unit ball. Motivated by numerous applications, we specialise the general framework to the case of Gaussian RCCSs, and we establish that the Gaussian structure of random sets is preserved under linear set–dynamics of random sets.

S. V. Raković is with St. Edmund Hall, Oxford University, UK. I. Matei is with the Institute for Research in Electronics and Applied Physics at the University of Maryland, College Park, MD 20742. J. S. Baras is with with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742, USA. E-mails:svr@sasavrakovic.com, imatei@umd.edu, and baras@umd.edu

Paper Structure: Section II provides the setting, necessary preliminaries and paper objectives. Section III considers the RCCSs, and derives the associated set–dynamics of the expected reach sets and the dynamics of the corresponding covariance functions. Section IV analyses the associated limiting behaviour, while Section V focuses on the case of Gaussian RCCSs. Section VI provides concluding remarks.

Basic Nomenclature and Definitions: The sets of non–negative, positive integers and nonnegative reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{R}_+$, respectively. For $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$ we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$ and write $\mathbb{N}_b$ for $\mathbb{N}_{[0:b]}$. Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X, \ y \in Y\}$, and we write $x \oplus X$ instead of $\{x\} \oplus X$. Given the sequence of sets $\{X_i \subset \mathbb{R}^n\}_{i=a}^{b}$, $a \in \mathbb{N}$, $b \in \mathbb{N}, b > a$, we denote $\bigoplus_{i=a}^{b} X_i := X_a \oplus \cdots \oplus X_b$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) the image of X under M is denoted by $MX := \{Mx \ : \ x \in X\}$. Given a matrix $M \in \mathbb{R}^{n \times n}$, $\rho(M)$ denotes the spectral radius of M, that is, the largest absolute value of its eigenvalues. A set $X \subset \mathbb{R}^n$ is *a C set* if it is compact (closed and bounded), convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is *a proper C set* if it is a C set and has non–empty interior. We say that a set $X \subseteq \mathbb{R}^n$ is a symmetric set w.r.t. $0 \in \mathbb{R}^n$ if $X = -X$. The collection of non–empty compact sets in $\mathbb{R}^n$ is denoted by $\mathrm{Com}(\mathbb{R}^n)$. The collection of non–empty compact, convex, sets in $\mathbb{R}^n$ is denoted by $\mathrm{ComConv}(\mathbb{R}^n)$. The convex hull of a set $X \subset \mathbb{R}^n$ is denoted by $\mathrm{co}(X)$. The support function $\mathrm{s}(X, \cdot)$ of a non–empty closed convex set $X \subset \mathbb{R}^n$ is given

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by

$$\mathrm{s}(X,y) := \sup_x \{y^T x \ : \ x \in X\} \text{ for } y \in \mathbb{R}^n.$$

Given a PC–set L in $\mathbb{R}^n$, the function $\mathrm{g}(L,\cdot)$ given by

$$\mathrm{g}(L,x) := \inf_\mu \{\mu \ : \ x \in \mu L, \ \mu \in \mathbb{R}_+\} \text{ for } x \in \mathbb{R}^n$$

is called the gauge (Minkowski) function of the set L. If L is a symmetric PC–set in $\mathbb{R}^n$, then $\mathrm{g}(L,\cdot)$ induces the vector norm $|x|_L := \mathrm{g}(L,x)$ whose unit norm ball is the set L. For $X \in \mathrm{Com}(\mathbb{R}^n)$ and $Y \in \mathrm{Com}(\mathbb{R}^n)$, the *Hausdorff distance (metric)* is given by

$$H(L,X,Y) := \min_\alpha \{\alpha : X \subseteq Y \oplus \alpha L, Y \subseteq X \oplus \alpha L, \alpha \geq 0\},$$

where L is a given, symmetric, proper C set in $\mathbb{R}^n$. The norm of a non–empty compact subset X of $\mathbb{R}^n$ (i.e., $X \in \mathrm{Com}(\mathbb{R}^n)$) is given by $\|X\|_L := H(L,X,\{0\})$.

II. Preliminaries

A. Setting

We consider the following autonomous discrete–time linear time–invariant (DLTI) system:

$$x^+ = Ax + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state and $w \in \mathbb{R}^n$ is an unknown but bounded disturbance. Thus, at any time $k \in \mathbb{N}$, the system (2.1) satisfies $x_{k+1} = Ax_k + w_k$. The unknown disturbance variable is bounded in the sense that, for all $k \in \mathbb{N}$, it holds that:

$$w_k \in W_k, \tag{2.2}$$

where, for each $k \in \mathbb{N}$, the disturbance set W_k is a random compact set (as defined in Section II-C). The sequence $\mathbf{W}_\infty := \{W_k\}_{k\in\mathbb{N}}$ is a sequence of independent, identically distributed (i.i.d.) random compact sets. The initial state of the system (2.1) belongs to a random compact set X_0:

$$x_0 \in X_0. \tag{2.3}$$

The random sets X_0 and $W_k, \ k \in \mathbb{N}$ are independent.

In view of the set–dynamics theoretical framework [6], similarly as in [7], we introduce the map $\mathcal{R}(\cdot,\cdot)$, given by:

$$\mathcal{R}(X,W) := AX \oplus W. \tag{2.4}$$

Clearly, the function $\mathcal{R}(\cdot,\cdot)$ maps $\mathrm{Com}(\mathbb{R}^n) \times \mathrm{Com}(\mathbb{R}^n)$ to $\mathrm{Com}(\mathbb{R}^n)$ as well as $\mathrm{ComConv}(\mathbb{R}^n) \times \mathrm{ComConv}(\mathbb{R}^n)$ to $\mathrm{ComConv}(\mathbb{R}^n)$. Our first standing assumption is:

Assumption 1: $A \in \mathbb{R}^{n\times n}$ is a strictly stable matrix.

As shown in [7], Assumption 1 implies that there exists a symmetric, proper C–set L in $\mathbb{R}^n$ and a scalar $\lambda \in [0,1)$ such that $AL \subseteq \lambda L$, so that for all $\mu \in \mathbb{R}_+$ and all $k \in \mathbb{N}$:

$$A^k \mu L \subseteq \lambda^k \mu L. \tag{2.5}$$

In light of this fact, we invoke a simplifying Assumption that is implied directly by Assumption 1:

Assumption 2: The set L utilized for the induced Hausdorff distance $H(L,\cdot,\cdot)$ is a symmetric, proper C–set L in $\mathbb{R}^n$ such that $AL \subseteq \lambda L$ for some fixed $\lambda \in [0,1)$.

The value of λ is, in fact, such that $\lambda \in [\rho(A),1)$. By (2.5), $\forall \ k \in \mathbb{N}$ and $\forall \ y \in \mathbb{R}^n$, we have $\mathrm{g}(L, A^k \mu y) \leq \lambda^k \mu \, \mathrm{g}(L,y)$ or, equivalently, $\mathrm{s}((A^T)^k \mu L^*, y) \leq \lambda^k \mu \, \mathrm{s}(L^*,y)$ where L^* is the dual set of the set L. Since the dual set of a symmetric proper C–set is itself a symmetric proper C–set [14], this fact yields directly a relevant consequence of Assumption 2:

Proposition 1: Suppose Assumption 2 holds. Then, for all $\mu \in \mathbb{R}_+$ and all $k \in \mathbb{N}$, it holds that:

$$\left(A^T\right)^k \mu L^* \subseteq \lambda^k \mu L^*, \tag{2.6}$$

where $L^* := \{y \in \mathbb{R}^n \ : \ \forall x \in L, \ y^T x \leq 1\}$ is the dual set of the set L, which is itself a symmetric, proper C–set in $\mathbb{R}^n$ and where $\lambda \in [0,1)$ is the scalar appearing in Assumption 2.

B. Reachability Analysis for the Deterministic Case

We now recall the standard reachability and invariance notions concerned with the classical case when X_0 and, for all $k \in \mathbb{N}$, $W_k = W$ are deterministic sets. In this standard setting (see, for example, [1]–[3] and references therein), reachability analysis reduces to the characterization of exact reach sets $X_k, \ k \in \mathbb{N}$ and exact reachable tube $\{X_k\}_{k\in\mathbb{N}}$ which are generated by deterministic set–dynamics:

$$X^+ = \mathcal{R}(X,W) \text{ so that}$$
$$\forall k \in \mathbb{N}, \ X_{k+1} = \mathcal{R}(X_k,W). \tag{2.7}$$

Thus, the reach set at time $k \in \mathbb{N}$ is the k^{th} iterate of the map $\mathcal{R}(\cdot,W) \ : \ \mathrm{Com}(\mathbb{R}^n) \to \mathrm{Com}(\mathbb{R}^n)$ evaluated at X_0, while the reachable tube is the trajectory of the system (2.7) with the initial condition equal to X_0. In this case, the reach sets admit an explicit representation given by:

$$\forall k \in \mathbb{N}_+, \ X_k := A^k X_0 \oplus \bigoplus_{i=0}^{k-1} A^i W. \tag{2.8}$$

When $\rho(A) < 1$ and W is a compact set, the reach sets X_k converge to the unique solution of the fixed point set equation [6], [7]:

$$X = \mathcal{R}(X,W), \text{ i.e., } X = AX \oplus W, \tag{2.9}$$

which is given explicitly by:

$$X_\infty := \bigoplus_{i=0}^{\infty} A^i W. \tag{2.10}$$

In fact, the set X_∞ is an exponentially stable attractor for the set–dynamics (2.7) with the basin of attraction being the whole space $\mathrm{Com}(\mathbb{R}^n)$ [6], [7].

Paper Objectives: In analogy with the deterministic setting, our main aims are to:

(i) Derive the reachability notions for the case where the set–dynamics (2.7) are driven by random sets and, in particular, derive the set–dynamics of the expectations of the corresponding reach sets as well as the dynamics of the associated covariance/variance functions; and
(ii) Specialize these notions to the case of Gausian RCCSs.

C. Random Sets – Technical Preliminaries

To clarify the use of random compact sets, we provide a brief overview of the necessary topological facts utilized throughout the manuscript (the interested reader is referred to [8]–[15] for a more detailed overview).

1) Topological Preliminaries: It is well–known that $\mathrm{Com}(\mathbb{R}^n)$ endowed with the Hausdorff distance $H(L,\cdot,\cdot)$ is a complete metric space [12], [14]. In fact, with the use of the Hausdorff distance $H(L,\cdot,\cdot)$ the space $\mathrm{Com}(\mathbb{R}^n)$ can be made into a separable, locally compact metric space [11], [12], [14]. It is also known that $\mathrm{ComConv}(\mathbb{R}^n)$ is a closed subset of $\mathrm{Com}(\mathbb{R}^n)$ and that the convex hull is a map $\mathrm{co}(\cdot) : \mathrm{Com}(\mathbb{R}^n) \to \mathrm{ComConv}(\mathbb{R}^n)$ which is continuous w.r.t. the Hausdorff distance $H(L,\cdot,\cdot)$ [11], [12], [14]. Additionally, as shown in [11], [12], [14], $\mathrm{ComConv}(\mathbb{R}^n)$ is an abstract, locally compact, convex cone which can be embedded isometrically into the Banach space $\mathcal{C}(L^*)$ of continuous functions on the dual unit ball $L^* := \{y \in \mathbb{R}^n : \forall x \in L,\ y^T x \le 1\}$ (w.r.t. the unit ball L) of $\mathbb{R}^n$ by identifying a set $X \in \mathrm{ComConv}(\mathbb{R}^n)$ with its support function:

$$\mathrm{s}(X,y) = \sup_x \{y^T x : x \in X\}, \tag{2.11}$$

for all $y \in L^*$. (We note that, here, one typically utilizes the Euclidean norm ball $L = \mathcal{B}_2 := \{x \in \mathbb{R}^n : x^T x \le 1\}$ so that its dual L^* satisfies $L^* = L$, or, one can utilize equivalently the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : x^T x = 1\}$.) It is well–known that this mapping preserves both the metric and linear structure [11], [14]. In view of the metric structure of $\mathrm{Com}(\mathbb{R}^n)$, measurability of a random set X can be taken in the Borel sense [9], [10], [12]. *Thus, a random set X (i.e., an almost surely non–empty random compact set) can be regarded as a measurable map defined on a probability space (Ω, Σ, P), and taking values in the collection $\mathrm{Com}(\mathbb{R}^n)$ of non–empty compact subsets of $\mathbb{R}^n$* [11], [13].

2) Expectation, Covariance and Variance Functions of a Random Set: We adapt the definition of the expectation of a random set as introduced by Artstein and Vitale [11]. *A selection of the random set X is a random vector x such that $x(\omega) \in X(\omega)$ holds almost surely.* We employ $E(\cdot)$ to denote expectation.

Definition 1: Let X be a random set such that each selection x has finite expectation $\mathrm{E}(x)$. The expectation of a random set X, denoted by $\mathrm{E}(X)$, is given by:

$$\mathrm{E}(X) := \{\mathrm{E}(x) : x \text{ is a selection of } X\}. \tag{2.12}$$

A necessary and sufficient condition for $\mathrm{E}(X)$ to be well–defined, (i.e., that $\mathrm{E}(X) \in \mathrm{Com}(\mathbb{R}^n)$), is that $\mathrm{E}(\|X\|_L) < \infty$ [11]. If the underlying probability space is nonatomic, then it is known [8], [10], [11] that $\mathrm{E}(\mathrm{co}(X)) = \mathrm{E}(X)$.

In light of the fact that the Minowski set addition is not, in general, an invertible operation, defining a covariance function of random set can be done by considering the covariance function of the support function associated with the random set (see [13], [15]). More precisely, for a random set X such that $\mathrm{E}(\|X\|_L^2) < \infty$, its covariance function, denoted by $\Gamma(X,\cdot,\cdot)$, is identified with the covariance function of the support function $\mathrm{s}(X,\cdot)$ considered to be a $\mathcal{C}(L^*)$–valued random variable, so that, for all $y \in L^*$ and all $z \in L^*$,:

$$\Gamma(X,y,z) := \mathrm{E}(\mathrm{s}(X,y)\,\mathrm{s}(X,z)) - \mathrm{E}(\mathrm{s}(X,y))\,\mathrm{E}(\mathrm{s}(X,z)). \tag{2.13}$$

By the same token, the variance function, denoted by $\Psi(X,\cdot)$ of a random set X such that $\mathrm{E}(\|X\|_L^2) < \infty$, is a function given, for all $y \in L^*$, by:

$$\Psi(X,y) := \Gamma(X,y,y). \tag{2.14}$$

As in the case with the expectation of a random set, the results utilized in the central limit theorem for random compact sets [13], [15] imply that the covariance function satisfies, for all $y \in L^*$ and all $z \in L^*$, that $\Gamma(X,y,z) = \Gamma(\mathrm{co}(X),y,z)$ while the associated variance function satisfies, for all $y \in L^*$, that $\Psi(X,y) = \Psi(\mathrm{co}(X),y)$.

Remark 1: In view of (2.13) and (2.14), we employ the covariance functions for analysis throughout this paper; however, all results verified for the covariance functions apply directly to the associated variance functions.

III. The Dynamics of the Expected Reach Sets and Covariance Functions

We first characterize the expectation and covariance function of a random set $MX \oplus Y$. A direct utilization of the strong law of large numbers for random compact sets established by Artstein and Vitale [11] implies that $\mathrm{E}(MX \oplus Y) = M\,\mathrm{E}(\mathrm{co}(X)) \oplus \mathrm{E}(\mathrm{co}(Y))$ as long as X and Y are independent random compact sets such that $\mathrm{E}(\|X\|_L) < \infty$ and $\mathrm{E}(\|Y\|_L) < \infty$. For any non–empty compact subset Z of $\mathbb{R}^n$ and any $u \in \mathbb{R}^n$ let $\mathrm{s}(Z,u) := \max_z\{u^T z : z \in Z\}$. With this in mind, for any two non–empty compact subsets of $\mathbb{R}^n$, say X and Y, and for any $u \in \mathbb{R}^n$, we have $\mathrm{s}(MX \oplus Y, u) = \mathrm{s}(X, M^T u) + \mathrm{s}(Y,u) = \mathrm{s}(\mathrm{co}(X), M^T u) + \mathrm{s}(\mathrm{co}(Y), u)$. Thus, a direct calculation verifies that the corresponding covariance function of the associated support function of $MX \oplus Y$ satisfies, for all $u \in L^*$ and all $v \in L^*$, that $\Gamma(Z,u,v) = \Gamma(\mathrm{co}(X), M^T u, M^T v) + \Gamma(\mathrm{co}(Y),u,v)$ as long as X and Y are independent random sets such that $\mathrm{E}(\|X\|_L^2) < \infty$ and $\mathrm{E}(\|Y\|_L^2) < \infty$. The discussion above yields directly a preliminary result utilized extensively throughout this paper:

Theorem 1: Fix any $M \in \mathbb{R}^{n\times n}$ such that $ML \subseteq L$ where L is the set used in the Hausdorff distance $H(L,\cdot,\cdot)$, and take any two independent random sets X and Y. Let

$$Z := MX \oplus Y. \tag{3.1}$$

(i) If $\mathrm{E}(\|X\|_L) < \infty$ and $\mathrm{E}(\|Y\|_L) < \infty$, then Z is a random set such that $\mathrm{E}(\|Z\|_L) < \infty$ and its expectation $\mathrm{E}(Z)$ satisfies:

$$\mathrm{E}(Z) = M\,\mathrm{E}(\mathrm{co}(X)) \oplus \mathrm{E}(\mathrm{co}(Y)), \tag{3.2}$$

(ii) If $\mathrm{E}(\|X\|_L^2) < \infty$ and $\mathrm{E}(\|Y\|_L^2) < \infty$, then Z is a random set such that $\mathrm{E}(\|Z\|_L^2) < \infty$ and its covariance function $\Gamma(Z,\cdot,\cdot)$ satisfies, for all $u \in L^*$ and $v \in L^*$:

$$\Gamma(Z,u,v) = \Gamma(\mathrm{co}(X), M^T u, M^T v) + \Gamma(\mathrm{co}(Y),u,v). \tag{3.3}$$

Returning to our setting, since X_0 and $W_k,\ k \in \mathbb{N}$ are random sets, the iteration of the map $\mathcal{R}(\cdot,\cdot)$:

$$\forall k \in \mathbb{N},\ X_{k+1} = \mathcal{R}(X_k, W_k) = AX_k \oplus W_k, \qquad (3.4)$$

generates random sets $X_k,\ k \in \mathbb{N}_+$.

Assumption 3: The assumptions on the random sets X_0 and $W_k,\ k \in \mathbb{N}$ are summarized by:

(i) The sets $W_k,\ k \in \mathbb{N}$ are i.i.d. RCCSs, which are identical copies of the RCCS W such that $\mathrm{E}(\|W\|_L^2) < \infty$ (which implies that $\mathrm{E}(\|W\|_L) < \infty$). Furthermore, the expectation $\mathrm{E}(W) = \overline{W} \in \mathrm{ComConv}(\mathbb{R}^n)$ and the covariance function $\Gamma(W,\cdot,\cdot)$ of the random set W are known so that, for all $k \in \mathbb{N}$,

$$\mathrm{E}(W_k) = \mathrm{E}(W) = \overline{W} \in \mathrm{ComConv}(\mathbb{R}^n) \qquad (3.5)$$

and, for all $y \in L^*$ and all $z \in L^*$,

$$\Gamma(W_k, y, z) = \Gamma(W, y, z). \qquad (3.6)$$

Additionally, for all $y \in L^*$ and all $z \in L^*$, it holds that $0 \le |\Gamma(W,y,z)| \le c_W |(y,z)|_{L^*\times L^*}$;

(ii) The set X_0 is a RCCS such that $\mathrm{E}(\|X_0\|_L^2) < \infty$ (which implies that $\mathrm{E}(\|X_0\|_L) < \infty$). Moreover, its expectation $\mathrm{E}(X_0)$ is known and it satisfies

$$\mathrm{E}(X_0) = \overline{X}_0 \in \mathrm{ComConv}(\mathbb{R}^n) \qquad (3.7)$$

while its covariance function $\Gamma(X_0,\cdot,\cdot)$ is also known and it satisfies, for all $y \in L^*$ and all $z \in L^*$, $0 \le |\Gamma(X_0,y,z)| \le c_{X_0} |(y,z)|_{L^*\times L^*}$; and

(iii) The random sets X_0 and $W_k,\ k \in \mathbb{N}$ are independent.

Remark 2: The RCCS X is a random set which is almost surely convex and compact. For simplicity, we omit the term "almost surely" whenever we refer to random sets which are almost surely convex and compact as no confusion should arise. The use of RCCSs is motivated merely by notational simplicity. In light of notions of Section II-C.2, the convexity requirement can be relaxed due to the "convexifying" effect of the Minkowski averaging [11], [13]; however, these generalizations are omitted due to page limitations.

We now discuss appropriate reachability notions when the random sets X_0 and $W_k,\ k \in \mathbb{N}$ satisfy Assumption 3. Assumption 3 ensures, by induction, that for all $k \in \mathbb{N}$, X_k and $W_{k+i},\ i \in \mathbb{N}$ are independent. Hence, Proposition 1 and Theorem 1 permit the use of set–dynamics of the associated expected reach sets accompanied with the dynamics of the corresponding covariance functions as stated by:

Proposition 2: Suppose Assumptions 2 and 3 hold and consider the sequence of random sets $\{X_k\}_{k\in\mathbb{N}}$ generated via (3.4). Then, for all $k \in \mathbb{N}_+$, X_k are RCCCs and satisfy $\mathrm{E}(\|X_k\|_L) < \infty$ and:

$$E(X_{k+1}) = AE(X_k) \oplus E(W_k). \qquad (3.8)$$

Furthermore, $\mathrm{E}(\|X_k\|_L^2) < \infty$ and, for all $y \in L^*$ and all $z \in L^*$, it holds that

$$\Gamma(X_{k+1}, y, z) = \Gamma(X_k, A^T y, A^T z) + \Gamma(W_k, y, z). \qquad (3.9)$$

A direct induction argument using Proposition 1, Theorem 1 and Proposition 2 allows us to characterize explicitly the expectations and covariance functions of the random sets $X_k,\ k \in \mathbb{N}$ generated via (3.4).

Proposition 3: Suppose Assumptions 2 and 3 hold and consider the sequence of random sets $\{X_k\}_{k\in\mathbb{N}}$ generated via (3.4). Then, for all $k \in \mathbb{N}_+$, (i) $\mathrm{E}(\|X_k\|_L) < \infty$ and the expectations $\overline{X}_k := E(X_k)$ of the RCCSs X_k satisfy:

$$\overline{X}_k = A^k \overline{X}_0 \oplus \bigoplus_{i=0}^{k-1} A^i \overline{W}, \qquad (3.10)$$

and (ii) $\mathrm{E}(\|X_k\|_L^2) < \infty$ and the covariances $\Gamma(X_k,\cdot,\cdot)$ of the RCCSs X_k satisfy, for all $y \in L^*$ and all $z \in L^*$,

$$\Gamma(X_k, y, z) = \Gamma(X_0, (A^T)^k y, (A^T)^k z) + \sum_{i=0}^{k-1} \Gamma(W, (A^T)^i y, (A^T)^i z). \qquad (3.11)$$

Clearly, in light of Propositions 2 and 3, the expected reach sets are fully characterized by deterministic set–dynamics (3.8) while the associated covariance functions evolve according to (3.9).

IV. The Limiting Behaviour

Section III motivates the analysis of the asymptotic behaviour of random sets X_k as $k \to \infty$ by verifying the existence and utilizing the properties of the associated fixed points. More precisely, as in the deterministic case [6], [7], we consider the fixed point set equation associated with the set–dynamics of the expected reach sets:

$$\mathrm{E}(X) = A\,\mathrm{E}(X) \oplus \mathrm{E}(W) \qquad (4.1)$$

and the associated functional fixed point equation (relevant for the dynamics of the corresponding covariance functions) given, for all $y \in L^*$ and all $z \in L^*$, by:

$$\Gamma(X, y, z) = \Gamma(X, A^T y, A^T z) + \Gamma(W, y, z). \qquad (4.2)$$

We now characterize a well–defined, RCCS such that (4.1) and (4.2) hold true. To this end, we consider the sequence of RCCSs defined, for all $k \in \mathbb{N}$, by:

$$\forall k \in \mathbb{N},\ S_k := \bigoplus_{i=0}^{k} R_i \text{ with } \forall i \in \mathbb{N},\ R_i := A^i W_i. \quad (4.3)$$

Under Assumption 3, for all $k \in \mathbb{N}$, R_k and S_k are independent RCCSs. A direct calculation reveals that, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}_+$, we have $H(L, S_{k+j}, S_k) \le H(L, \bigoplus_{i=k+1}^{k+j} R_i, \{0\})$. Thus, by (2.5), we have:

$$H(L, S_{k+j}, S_k) \le \sum_{i=k+1}^{k+j} \lambda^i \|W_i\|_L. \qquad (4.4)$$

Hence, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}_+$, we have:

$$\mathrm{E}(H(L, S_{k+j}, S_k)) \le \sum_{i=k+1}^{k+j} \lambda^i\, \mathrm{E}(\|W_i\|_L). \qquad (4.5)$$

Now, for all $i \in \mathbb{N}$, $\mathrm{E}(\|W_i\|_L) = \mathrm{E}(\|W\|_L) < \infty$, and, hence, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}_+$, it holds that

$$\mathrm{E}(H(L, S_{k+j}, S_k)) \le \lambda^{k+1}(1-\lambda)^{-1}\, \mathrm{E}(\|W\|_L) \qquad (4.6)$$

Relation (4.6) implies that, for any $j \in \mathbb{N}_+$, the scalar sequence $\{\mathrm{E}(H(L, S_{k+j}, S_k))\}_{k\in\mathbb{N}}$ converges to 0 as $k \to \infty$, and therefore the sequence $\{S_k\}_{k\in\mathbb{N}}$ is Cauchy in the mean sense. By completeness, the sequence $\{S_k\}_{k\in\mathbb{N}}$ converges in the mean sense, and consequently, it converges in probability and distribution. Furthermore, by Markov inequality, for any $\varepsilon > 0$ we have that the probability of $H(L, S_{k+j}, S_k)$ being strictly bigger than ε is less than $\frac{E(H(L,S_{k+j},S_k))}{\varepsilon}$. But, since the scalar sequence $\{\mathrm{E}(H(L, S_{k+j}, S_k))\}_{k\in\mathbb{N}}$ converges to zero geometrically, a direct use of the Borel–Cantelli Lemma [17] implies that the sequence $\{S_k\}_{k\in\mathbb{N}}$ is Cauchy almost surely. Thus, by completeness, it follows that the sequence of RCCSs $\{S_k\}_{k\in\mathbb{N}}$ converges almost surely to some RCCS S_∞, which we denote by

$$S_\infty := \bigoplus_{i=0}^{\infty} A^i W_i. \tag{4.7}$$

Summarizing, we have established:

Theorem 2: Suppose Assumptions 2 and 3 hold. Then the sequence of random sets $\{S_k\}_{k\in\mathbb{N}}$ specified by (4.3) converges almost surely to a well–defined RCCS S_∞ specified via (4.7).

The expectation $\overline{S}_\infty := \mathrm{E}(S_\infty)$ of the random set S_∞ satisfies, under Assumptions 2 and 3, that $\overline{S}_\infty = \mathrm{E}(\bigoplus_{i=0}^{\infty} A^i W_i) = \bigoplus_{i=0}^{\infty} A^i\,\mathrm{E}(W_i) = \bigoplus_{i=0}^{\infty} A^i \overline{W}$ so that

$$\overline{S}_\infty = A\overline{S}_\infty \oplus \overline{W}. \tag{4.8}$$

Under Assumptions 2 and 3, the sequence of functions $\{\sum_{i=0}^{k} \Gamma(A^i W, \cdot, \cdot)\}_{k\in\mathbb{N}}$ is a sequence of continuous functions which satisfies, for all $y \in L^*$ and all $z \in L^*$, $\sum_{i=0}^{k} \Gamma(A^i W, y, z) = \sum_{i=0}^{k} \Gamma(W, (A^T)^i y, (A^T)^i z)$. Furthermore, Assumptions 2 and 3 guarantee that $\{\sum_{i=0}^{k} \Gamma(A^i W, \cdot, \cdot)\}_{k\in\mathbb{N}}$ is a Cauchy sequence of continuous functions which converges uniformly on a compact set $L^* \times L^*$ to a continuous function denoted by $\sum_{i=0}^{\infty} \Gamma(A^i W, \cdot, \cdot)$. By inspection, the limit function $\sum_{i=0}^{\infty} \Gamma(A^i W, \cdot, \cdot)$ satisfies, for all $y \in L^*$ and all $z \in L^*$, that $\sum_{i=0}^{\infty} \Gamma(A^i W, y, z) = \sum_{i=0}^{\infty} \Gamma(W, (A^T)^i y, (A^T)^i z)$. Now, a direct calculation also shows that, for all $y \in L^*$ and all $z \in L^*$, $\Gamma(S_\infty, y, z) = \sum_{i=0}^{\infty} \Gamma(W, (A^T)^i y, (A^T)^i z)$ so that, for all $y \in L^*$ and all $z \in L^*$, it holds that:

$$\Gamma(S_\infty, y, z) = \sum_{i=0}^{\infty} \Gamma(W, (A^T)^i y, (A^T)^i z). \tag{4.9}$$

Furthermore $\Gamma(AS_\infty, y, z) = \Gamma(S_\infty, A^T y, A^T z)$ so that, for all $y \in L^*$ and all $z \in L^*$, we have:

$$\Gamma(S_\infty, A^T y, A^T z) = \sum_{i=1}^{\infty} \Gamma(W, (A^T)^i y, (A^T)^i z). \tag{4.10}$$

It follows that, for all $y \in L^*$ and all $z \in L^*$, we have:

$$\Gamma(S_\infty, y, z) = \Gamma(S_\infty, A^T y, A^T z) + \Gamma(W, y, z). \tag{4.11}$$

Consequently, we have established:

Theorem 3: Suppose Assumptions 2 and 3 hold. Then the RCCS S_∞ specified via (4.7) satisfies the fixed point relations (4.1) and (4.2).

Theorems 2 and 3 allows us to discuss convergence of the sequence of RCCSs $\{X_k\}_{k\in\mathbb{N}}$ generated by (3.4). To this end, consider the sequence of random sets $\{\tilde{S}_k\}_{k\in\mathbb{N}}$ where, for all $k \in \mathbb{N}$,

$$\tilde{S}_{k+1} := A^{k+1} X_0 \oplus S_k \text{ with } \tilde{S}_0 := X_0, \tag{4.12}$$

and where random sets $S_k,\ k \in \mathbb{N}$ are specified as in (4.3). By construction, for all $k \in \mathbb{N}$, the sets $\tilde{S}_k$ of (4.12) are RCCSs. Since $H(L, \tilde{S}_{k+1}, S_k) \le \lambda^{k+1} \|X_0\|_L$ it follows that $\mathrm{E}(H(L, \tilde{S}_{k+1}, S_k)) \le \lambda^{k+1}\,\mathrm{E}(\|X_0\|_L)$ and, consequently, the scalar sequence $\{\mathrm{E}(H(L, \tilde{S}_{k+1}, S_k))\}_{k\in\mathbb{N}}$ converges to 0. Thus, similarly as discussed above Theorem 2, the sequence of RCCSs $\{\tilde{S}_k\}_{k\in\mathbb{N}}$ converges almost surely to S_∞ specified via (4.7). But $\tilde{S}_0 = X_0$ and, for all $k \in \mathbb{N}_+$, $\tilde{S}_k = A^k X_0 \oplus \bigoplus_{i=0}^{k-1} A^i W_i$ and $X_k = A^k X_0 \oplus \bigoplus_{i=0}^{k-1} A^{k-i} W_i$ are identical due to Assumption 3. Hence, since $\{\tilde{S}_k\}_{k\in\mathbb{N}}$ converges almost surely to S_∞, it follows that $\{X_k\}_{k\in\mathbb{N}}$ converges in distribution to the same limit S_∞ as stated by:

Theorem 4: Suppose Assumptions 2 and 3 hold. Then the sequence of RCCS $\{X_k\}_{k\in\mathbb{N}}$ generated by (3.4) converges in distribution to the RCCS S_∞ specified via (4.7).

V. Gaussian Random Convex Compact Sets

We now focus on the case of Gaussian RCCSs. By [15, Theorem 2.11, Section 3], a RCCS X is Gaussian if and only if it takes the form:

$$X = \overline{X} \oplus x, \tag{5.1}$$

where $\overline{X}$ is a deterministic non–empty convex compact subset of $\mathbb{R}^n$ and x is a Gaussian random vector in $\mathbb{R}^n$. Hence, in this section, we replace Assumption 3 with:

Assumption 4: (i) The sets $W_k := \overline{W} + w_k,\ k \in \mathbb{N}$ are i.i.d. Gaussian RCCSs. Furthermore, for all $k \in \mathbb{N}$, $\overline{W} \in \mathrm{ComConv}(\mathbb{R}^n)$ is a known deterministic set and w_k is a Gaussian random vector with $\mathrm{E}(w_k) = \overline{w} \in \mathbb{R}^n$ and $\mathrm{E}((w_k - \overline{w})(w_k - \overline{w})^T) = R$ for some $R \in \mathbb{R}^{n\times n}$ with $R = R^T \ge 0$;

(ii) The set $X_0 := \overline{X}_0 + x_0$ is a Gaussian RCCS, i.e., $\overline{X_0} \in \mathrm{ComConv}(\mathbb{R}^n)$ is a known deterministic set and x_0 is a Gaussian random vector with $\mathrm{E}(x_0) = \overline{x}_0 \in \mathbb{R}^n$ and $\mathrm{E}((x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T) = Q_0$ for some $Q_0 \in \mathbb{R}^{n\times n}$ with $Q_0 = Q_0^T \ge 0$; and

(iii) The random sets X_0 and $W_k,\ k \in \mathbb{N}$, or equivalently, the random vectors x_0 and $w_k,\ k \in \mathbb{N}$ are independent.

Now, suppose that, for some $k \in \mathbb{N}$, the random set X_k generated via set–dynamics (3.4) is a Gaussian RCCS so that $X_k = \overline{X}_k \oplus x_k$ where $\overline{X}_k$ is a deterministic convex compact subset of $\mathbb{R}^n$ and x_k is a Gaussian random vector in $\mathbb{R}^n$. Then, by (3.4), it follows that:

$$\begin{aligned} X_{k+1} &= AX_k \oplus W_k = A\overline{X}_k \oplus \overline{W} \oplus (Ax_k + w_k) \\ &= \overline{X}_{k+1} \oplus x_{k+1}, \end{aligned} \tag{5.2}$$

where clearly $\overline{X}_{k+1} := A\overline{X}_k \oplus \overline{W}$ is a deterministic convex compact subset of $\mathbb{R}^n$ and $x_{k+1} = Ax_k + w_k$ is a Gaussian random vector in $\mathbb{R}^n$. Hence, under Assumption 4, the set–dynamics (3.4) generates the sequence $\{X_k\}_{k\in\mathbb{N}}$ of Gaussian

RCCSs. A direct calculation by using (3.3) and (3.4) reveals that the expectations $\mathrm{E}(X_k)$ and covariance functions $\Gamma(X_k,\cdot,\cdot)$ of the corresponding Gaussian RCCSs $X_k,\ k\in\mathbb{N}$ are generated, for all $k\in\mathbb{N}$, by:

$$\begin{aligned}&\mathrm{E}(X_k)=\overline{X}_k\oplus\overline{x}_k,\ \text{where},\\ &\overline{X}_{k+1}=A\overline{X}_k\oplus\overline{W}\ \text{and}\ \overline{x}_{k+1}=A\overline{x}_k+\overline{w},\end{aligned}\qquad(5.3)$$

and, for all $y\in L^*$ and all $z\in L^*$,

$$\Gamma(X_k,y,z)=y^TQ_kz\ \text{with}\ Q_{k+1}=AQ_kA^T+R.\qquad(5.4)$$

An analogue of Proposition 3 now easily follows:

Proposition 4: Suppose Assumptions 2 and 4 hold. Consider the sequence of random sets $\{X_k\}_{k\in\mathbb{N}}$ generated via (3.4). Then, for all $k\in\mathbb{N}_+$, (i) X_k is a Gaussian RCCS, (ii) The expectation $\mathrm{E}(X_k)$ satisfies:

$$\mathrm{E}(X_k)=\left(A^k\overline{X}_0\oplus\bigoplus_{i=0}^{k-1}A^i\overline{W}\right)\oplus\left(A^k\overline{x}_0+\sum_{i=0}^{k-1}A^i\overline{w}\right),\qquad(5.5)$$

and, (iii) The covariance $\Gamma(X_k,\cdot,\cdot)$ satisfies, for all $y\in L^*$ and all $z\in L^*$, that:

$$\begin{aligned}&\Gamma(X_k,y,z)=y^TQ_kz,\ \text{where}\\ &Q_k=A^kQ_0(A^T)^k+\sum_{i=0}^{k-1}\left(A^iR(A^T)^i\right)\end{aligned}\qquad(5.6)$$

with $Q_k=Q_k^T\geq 0$.

Assumptions 2 and 4 also ensure that the random set S_∞, as specified via (4.7), is a well–defined, Gaussian RCCS. In particular, its expectation $\mathrm{E}(S_\infty)$ satisfies:

$$\mathrm{E}(S_\infty)=\left(\bigoplus_{i=0}^{\infty}A^i\overline{W}\right)\oplus\left((I-A)^{-1}\overline{w}\right),\qquad(5.7)$$

while its covariance function $\Gamma(S_\infty,\cdot,\cdot)$ satisfies, for all $y\in L^*$ and all $z\in L^*$,

$$\Gamma(S_\infty,y,z)=y^TQ_\infty z,\ Q_\infty:=\sum_{i=0}^{\infty}\left(A^iR(A^T)^i\right)\qquad(5.8)$$

with $Q_\infty=Q_\infty^T\geq 0$ and where the sum $\sum_{i=0}^{\infty}\left(A^iR(A^T)^i\right)$ is well–defined and symmetric positive definite since $\rho(A)<1$ and $R=R^T\geq 0$. The Gaussian nature of RCCSs, in conjunction with the facts indicated above, yields directly a slight generalization of Theorems 3 and 4:

Theorem 5: Suppose Assumptions 2 and 4 hold. Consider the sequence of random sets $\{X_k\}_{k\in\mathbb{N}}$ generated by (3.4) (or, equivalently by (5.2)) and the random set S_∞ given via (4.7). Then: (i) the random set S_∞ is a Gaussian RCCS with the expectation $\mathrm{E}(S_\infty)$ and covariance function $\Gamma(S_\infty,\cdot,\cdot)$ specified via (5.7) and (5.8), respectively; (ii) The Gaussian RCCS S_∞ satisfies the fixed point relations in (4.1) and (4.2); and (iii) The sequence of random sets $\{X_k\}_{k\in\mathbb{N}}$ is a sequence of Gaussian RCCSs which converges almost surely to the Gaussian RCCS S_∞.

VI. Concluding Remarks

We have studied linear set–dynamics driven by RCCSs, and we have established that the expected reach sets evolve according to deterministic linear set–dynamics while the corresponding dynamics of covariance functions evolves on the Banach space of continuous functions on the dual unit ball. We have also specialised the general framework to the case of Gaussian RCCSs, and we have shown that Gaussian structure of random sets is preserved under linear set–dynamics of random sets.

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References

[1] F. C. Schweppe, *Uncertain Dynamic Systems*. Englewood Cliffs, NJ: Prentice Hall, 1973.

[2] A. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*, ser. Systems & Control: Foundations & Applications. Boston, Basel, Berlin: Birkhauser, 1997.

[3] S. V. Raković and M. Fiacchini, "Approximate Reachability Analysis for Linear Discrete Time Systems Using Homothety and Invariance," in *Proceedings of the 17th IFAC World Congress IFAC 2008*, Seoul, Korea, 2008.

[4] J. P. Aubin, *Viability theory*, ser. Systems & Control: Foundations & Applications. Boston, Basel, Berlin: Birkhauser, 1991.

[5] F. Blanchini and S. Miani, *Set–Theoretic Methods in Control*, ser. Systems & Control: Foundations & Applications. Boston, Basel, Berlin: Birkhäuser, 2008.

[6] Z. Artstein and S. V. Raković, "Feedback and Invariance under Uncertainty via Set Iterates," *Automatica*, vol. 44, no. 2, pp. 520–525, 2008.

[7] S. V. Raković, "Minkowski Algebra and Banach Contraction Principle in Set Invariance for Linear Discrete Time Systems," in *Proceedings of 46th IEEE Conference on Decision and Control, CDC 2007*, New Orleans, LA, USA, December 2007.

[8] R. J. Aumann, "Integrals of Set–Valued Functions," *J. Math. Anal. Appl.*, vol. 12, pp. 1–12, 1965.

[9] P. Billingsley, *Convergence of Probability Measures*. New York: Wiley, 1968.

[10] Z. Artstein, "On the Calculus of Set–Valued Functions," *Indiana Univ. Math. J.*, vol. 24, pp. 433–441, 1974.

[11] Z. Artstein and R. A. Vitale, "A Strong Law of Large Numbers for Random Compact Sets," *The Annals of Probability*, vol. 3, no. 5, pp. 879–882, 1975.

[12] G. Matheron, *Random Sets and Integral Geometry*. New York: Wiley, 1975.

[13] W. Weil, "An Application of the Central Limit Theorem for Banach–space–valued Random Variables to the Theory of Random Sets," *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 60, pp. 203–208, 1982.

[14] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*. Cambridge, England: Cambridge University Press, 1993, vol. 44, of Encyclopedia of Mathematics and its Applications.

[15] I. Molchanov, *Theory of Random Sets (Probability and Its Applications)*. London Limited: Springer–Verlag, 2005.

[16] Z. Artstein and S. V. Raković, "Set Invariance Under Output Feedback : A Set–Dynamics Approach," *International Journal of Systems Science*, vol. 42, no. 4, pp. 539–555, 2011.

[17] G. Grimmet and D. Stirzaker, *Probability and Random Processes*, third edition ed. Oxford University Press, 2001.