Optimal Event-Triggered Control of Nondeterministic Linear Systems

Dipankar Maity and John S. Baras

***Abstract*—We consider an event-triggered controller synthesis problem to replace the continuous feedback policy with an intermittent feedback policy for a nondeterministic linear system. An event-triggered framework communicates the measurement to the controller only at certain discrete time instances which are generated by an event generator. The objective of this paper is to synthesize an optimal-event generator and controller pair such that the state trajectory of the event-triggered system mimics that of the feedback system with arbitrary precision. The optimality is in the sense that the least number of state measurements are sent to the controller in order to compute the control signal. The results of this paper show that such an optimal event-triggered controller retains the linear structure when the continuous feedback controller is linear; and the optimal event generator follows a threshold-based policy, where the event generator decides to send the state measurement to the controller every time a certain signal exceeds that threshold. Finally, the similar framework was extended for a controller synthesis of infinite horizon.The structural properties of the optimal event-triggered controller and event generator remain unchanged when extended to an infinite horizon.**

***Index Terms*—Event-triggered control, intermittent feedback, linear feedback control systems, open loop systems, optimal control.**

I. Introduction

CONSIDER the generic linear nondeterministic control system evolving in $\mathbb{R}^n$ as given in the following equation (1):

$$\dot{x} = Ax + Bu + d$$
$$x(0) = x_0, \tag{1}$$

where the initial condition $x(0)$ is known, and the nondeterminism arises due to the disturbance signal $d(t)$. Given a control law $u(t) = K(t, x(t))$, control of such a system requires continuous monitoring of the sensor measurements, and continuous transmission of the sensed measurements to the controller. Thus, sensing, communication, and computing are integrated in an inseparable way. In a centralized system, the performance depends on the continuity of the communication, and computing the control signal accurately. In a distributed system, although the controller is implemented distributively, however, it also requires continuous interactions among the subsystems. Sensing, communication, and data handling are indispensable parts for networked control systems. As a result, their performance is generally determined by the available resources to perform sensing, transmission, and computation. Scarcity of such resources are the sources of performance degradation for large and interconnected systems.

In the recent past, researchers have proposed novel techniques to approximate the control law $u = K(t, x(t))$, for system (1), in such a way that require only a finite number of transmissions (i.e., discrete-time transmissions) to overcome the requirement of continuous sensing, continuous transmitting, and continuous computing [1]–[3]. Event-based (Event-triggered) control has been proved to be remarkably effective in dealing with limited resources such as transmission bandwidth, sensing energy, and computational resources. Clearly, such an approximate control signal will only lead to a behavior of the state trajectory, which is approximate to the trajectory obtained from continuous feedback. Such a control scheme generally has two components: the *Controller* and the *Event-Generator*. The Event-generator decides the discrete time instances when the state measurement is to be transmitted, and the controller computes the control signal based on the received measurements from the event-generator.

A great deal of research has been performed in the last few decades to improve such frameworks, and extend them to nonlinear and stochastic systems. In [4], a comparison between the performance of event-based control and periodic-sampling-based control has shown that under some conditions the event-based control performs better than the periodic control. A simple proportional-integral-derivative (PID) controller is proposed in [3] for event-based control that reduces large CPU computation at the cost of minor control performance degradation. The supremacy of event-based strategy over a periodic-sampling strategy is not only in reducing the number of transmission when there is not much variation in the measurements, but also in increased transmission when there is rapid variation. In periodic sampling, the challenge is to find the suitable period of transmission to guarantee a certain level of performance. In the recent studies, the foci mostly have been on finding a feasible

Manuscript received February 23, 2018; revised March 7, 2018, December 16, 2018, and February 20, 2019; accepted April 5, 2019. Date of publication May 9, 2019; date of current version January 28, 2020. This work was supported in part by the U.S. Air Force Office of Scientific Research MURI under Grant FA9550-09-1-0538, in part by the DARPA through ARO under Grant W911NF1410384, in part by the National Science Foundation under Grant CNS-1035655, and in part by the National Institute of Standards and Technology under Grant 70NANB11H148. Recommended by Associate Editor Q.-S. Jia. *(Corresponding author: John S. Baras.)*

The authors are with the Department of Electrical and Computer Engineering, Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (e-mail: dmaity@umd.edu; baras@umd.edu).

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Digital Object Identifier 10.1109/TAC.2019.2915748

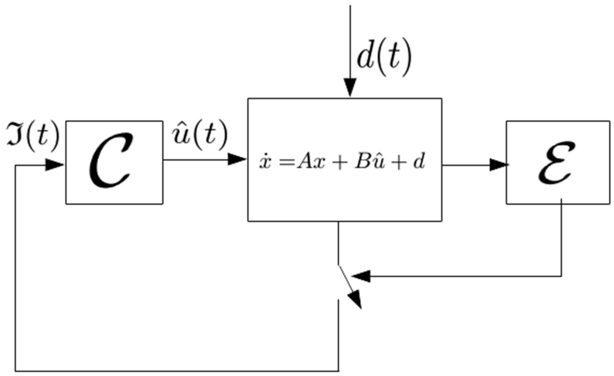

Fig. 1. Event-based control loop with three subsystems: control input generator, the plant, and the event-generator. The switched communication link allows communication from $\mathcal{E}$ to $\mathcal{C}$ only in a discrete-time manner.

controller and a compatible event-generator that together can approximate the continuous feedback trajectory with an arbitrary precision. The aim of this paper is to identify the optimal controller and the event-generator pair, which minimizes the total number of measurement transmissions, hence entailing minimal energy, bandwidth, and computation resources.

In event-based control, self-triggered control, or periodic control, the controller being unable to access the continuous state, it estimates the state, and the estimated state is used to produce the control input. Since the generated control input is different from the actual (continuous) feedback input, the response of the system is not as it would have been if there were a continuous state feedback. Let the state of the continuous feedback system be denoted as $x_c(t)$ and the state of the event-based system as $x(t)$. The signal $e(t) = x(t) - x_c(t)$ denotes the deviation in trajectories for using the event-based controller. For a given $\epsilon > 0$, the aim is to find an event-generator and a corresponding controller such that $\|e(t)\| \leq \epsilon$ for all t while the number of state measurement transmissions is minimized. The framework of our work is similar to [5] and [6], but none of them addresses the question of optimality of the number of transmitted measurements. The framework is schematically represented in Fig. 1 where the event generator ($\mathcal{E}$) determines the triggering instances and consequently sends the state information to the controller through the switched communication link. The system is influenced by the exogenous disturbance $d(t)$. In [5], for a similar problem (without the optimality criterion) it was assumed that the closed-loop plant dynamics with linear feedback is asymptotically stable (i.e., $A - BK$ is Hurwitz). In this paper, no such assumption is made. Similar to [5], we restrict ourselves to design event-based controller to replace the continuous controller, which are linear feedback, i.e., $K(t, x(t)) = K(t)x(t)$. If the system is not asymptotically stable, the residual error $e(t_k)$ after the kth triggering persists, and for unstable systems it might increase exponentially. The proposed approach assures that the control input can be designed (by introducing an "corrective" control component $\psi(t)$) in a way that can mitigate this residual error.

The main contributions of this paper are as follows.

First, the optimal structure of the event-triggered controller is derived and it is found to be only dependent on the latest state information, and not on all the prior measurements. Thus, the controller does not need any (extra) memory (except latest measurement) to implement the control law. Furthermore, it is found that the optimal controller is linear with respect to the latest state measurement. Moreover, we show uniqueness (and existence) of the optimal controller.

Second, we show that the controllability of the system is sufficient to ensure (by constructing an additional corrective control) that there exists an event-triggered controller and an event-generator so that the norm of the error $\|e(t)\|$ can be bounded by any given positive constant ϵ for all t. Thus, our approach is applicable to those systems where the closed-loop system is not Hurwitz; hence extending the applicability of event-triggered controllers to the class of problems that are not readily handled by existing techniques such as [5].

Third, we design our event-triggering mechanism that minimizes the total number of triggering under the worst case disturbance ($d(\cdot)$). It is shown by the study that such an event-triggering mechanism has a threshold-based policy. This policy is found by solving a certain dynamic programming problem, and the policy is unique. Such a triggering policy does not exhibit Zeno behavior, i.e., inter-triggering duration has a finite positive lower bound.

The rest of this paper is organized as follows. Section I-A performs a comprehensive literature survey; Section III formulates the general problem that will be addressed; Section IV provides the optimal controller synthesis; Section V describes the optimal event-triggering strategy; and Section VII illustrates the framework with examples. Finally, we conclude our work with a discussion in Section VIII.

A. Literature Review

Event-based controller synthesis is a well studied topic in control for more than a decade, and the literature is vast and enriched with various aspects of event-based frameworks on many systems. This section provides a brief and yet a concise representation of the related works.

In the past, various seemingly similar frameworks have been studied to reduce the communication overhead, e.g., event-based control [7], self-triggered control [8], [9] and periodic-time control [10], [11], [12]. The essence behind these techniques is to transmit measurements at discrete time instances rather than communicating continuously. Even comparisons among such methodologies have been performed to judge their effectiveness, see for example [4], [7]. In the literature, asynchronous control [13], event-based sampling [1], event-driven sampling [2], Lebesgue sampling [7], deadband sampling [14] have been proposed to carry out the idea of supplicating communication only when some event has been occurred. In [15], the authors analyzed event-based control in a stochastic setting. The works of [16]–[18] took a different approach by reducing the information content rather than reducing the communication frequency. A state feedback approach for an event-based system is considered in [5] where the feedback control is generated from another system, which is updated every time a trigger is introduced. Later, this framework was extended to consider the output feedback scenario in [19]. Output feedback based decentralized event-triggered control has also been considered in [20] and [21]. Zhang *et al.* [21] studied the problem of multiagent consensus in an output feedback event-based framework. Event-based control for distributed interconnected linear systems is proposed in [22] and [23].

In event-triggered framework, there are some works which directly aim to optimize certain cost function rather than penalizing the actual trajectory deviation. Such an event-based control of the standard LQG problem in a lossy channel is studied in [24], which also proposed a suboptimal solution was proposed. An event-triggered state estimation has been the focus of study in [25]. For a scalar system, a probabilistic triggering strategy is studied in [26].

In parallel, event-based control for nonlinear systems have been explored in the recent past. Asymptotic stability of an event-triggered nonlinear system is studied in [27] with the assumption of input-to-state stability. Tabuada [28] studied a real-time scheduling and stabilizing control task under event-triggered framework. This paper is extended for homogeneous and polynomial nonlinear systems in [29]. A feedback linearization based approach was taken in [30] to design an event-based controller for nonlinear systems. The nonlinear counterpart of [6] is investigated in [31] using a Lyapunov function based approach. An Lyapunov function based approach was also taken in [32] to study nonlinear event-based systems under delay and packet dropouts.

II. Notation

$x(t)$: state of the event-triggered system at time t, $x_c(t)$: state of the corresponding continuous feedback system at t, $e(t) = x(t) - x_c(t)$: error in the state trajectory. $d : \mathbb{R}_+ \to \mathbb{R}^n$: the exogenous disturbance, t_i: the ith triggering instance, $x(t_i)$: value of the state at the ith triggering instance, $\theta(t)$: the latest triggering instance before time t, $N(t)$: number of measurements sent until time t, $\mathfrak{I}(t) = \{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)}$ (or $\{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)} \cup \{e(t_i)\}_{i=1}^{N(t)}$): the information available to controller at time t, $\mathcal{K}(t, \mathfrak{I}(t))$: the generic structure of the controller, $\mathcal{C}$: event-triggered controller, $\mathcal{E}$: event generator, $u(t)$: continuous feedback input, $\hat{u}(t)$: event-triggered (intermittent feedback) input, J: cost function of total number of transmissions, $\Phi(t, s)$, $\tilde{\Phi}(t, s)$: state transition matrices, $\|\cdot\|$: a norm in $\mathbb{R}^n$.

To maintain brevity, throughout this paper, we will suppress the argument(s) of the functions, e.g., $x(t)$ will be denoted as x, $d(t)$ as d, etc.

III. Problem Formulation

Let us consider the linear nondeterministic dynamics of a system to be given as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu + d \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where for all t, $x(t) \in \mathbb{R}^n$ is the state of the system and $u(t)$ is the control. $d(t)$ is an n dimensional exogenous disturbance to the system, which is nondeterministic. We assume that $d : [0, \infty) \to \mathbb{R}^n$ is a Lebesgue integrable function in $\mathcal{L}^1([0, \infty))$ and consequently (2) has a well defined solution for all $t \geq 0$. When restricting ourselves to a finite horizon, we will consider $d(\cdot) \in \mathcal{L}^1([0, T])$. Some special remarks will be made when $d \in \mathcal{L}^\infty([0, T])$ or $d \in \mathcal{L}^\infty([0, \infty))$ In what follows, we will denote $d(\cdot) \in \mathcal{D}$ where $\mathcal{D} = \mathcal{L}^1([0, T])$ for a finite horizon problem and $\mathcal{D} = \mathcal{L}^1([0, \infty])$ for an infinite horizon problem. Sometimes we will consider $\mathcal{D} = \mathcal{L}^\infty([0, T])$ or $\mathcal{L}^\infty([0, \infty])$ for making certain remarks, and $\mathcal{D}$ will be explicitly mentioned whenever we do so.

Let us consider a feedback control $u(t) = -Kx(t)$ that has been designed to achieve some desirable behavior on the trajectory $x(\cdot)$. In the absence of d, it is sufficient to know only the initial state x_0 to calculate $u(t)$ for all t; however, the presence of d makes it absolutely necessary to know $x(t)$ in order to calculate $u(t)$ precisely.

The closed-loop continuous feedback system has the state dynamics

$$\dot{x}_c = \tilde{A}x_c + d \tag{3}$$

where $\tilde{A} = A - BK$.

The communication of the state measurement to the controller is done in a discrete-time manner and on demand. Due to the availability of discrete measurements, $\{x(t_i)\}_{i \in \mathbb{N}}$, as opposed to continuous measurements $\{x(s)\}_{s \geq 0}$, the computation of the control $u(t)$ will not be accurate, and hence, the trajectory $x(\cdot)$ will deviate from its desired trajectory $x_c(\cdot)$. In this paper, given an $\epsilon > 0$, our constraint on the controller $\mathcal{C}$ and event-generator $\mathcal{E}$ is to ensure $\|x(t) - x_c(t)\| \leq \epsilon$ for all t and for all realization of the disturbance $d(\cdot)$.

In this framework, since the communication is done in a discrete-time manner, the exact state of the system, $x(t)$ is available to the controller only at those time instances t_i. Let at any time t, $\theta(t)$ denote the latest instance ($t_i \leq t$) when the state value ($x(t_i)$) was communicated to the controller. $\theta(t) = 0$ for all $t < t_1$ where t_1 is the first triggering instance. Thus, $\theta(t)$ is a piecewise constant function and $\theta(t) \leq t$ where the equality holds at the triggering instances. Moreover, $\frac{d\theta(t)}{dt} = 0$ for all $t \neq \theta(t)$.

The objective is to design the control in a way such that it does not require the continuous measurement of the state of the system and, nonetheless, it drives the new system to approximate the closed-loop system (3) within the given tolerance bound for any realization of the disturbance d. Let $\hat{u}(t) = \mathcal{K}(t, x_0 \cup \{x(t_i)\}_{i=1}^{N(t)})$ where $t_{N(t)} = \theta(t)$; $N(t)$ denotes the total number of measurements sent until time t. The new system with $\hat{u}$ as control input has the dynamics

$$\begin{aligned} \dot{x} &= Ax + B\hat{u} + d \\ x(0) &= x_0. \end{aligned} \tag{4}$$

The deviation of $x(t)$ from $x_c(t)$ will depend on the choice of $\{t_i\}_{i \in \mathbb{N}}$, $N(t)$, and $\mathcal{K}(\cdot, \cdot)$; and these are our optimization variables. We divide these variables into two groups $\mathcal{E} = \{N(\cdot), \{t_i\}_{i=1}^{N(\cdot)}\}$, and $\mathcal{C} = \mathcal{K}(\cdot, \cdot)$, where $\mathcal{E}$ will be referred as the event-generator, which will decide the sampling instances $\{t_i\}_{i=1}^{N(t)}$ and send $x(t_i)$ to the controller; and $\mathcal{C}$ will be named as the event-triggered controller, which will generate the input $\hat{u}(t)$ based on the measurements sent by $\mathcal{E}$.

Formally, we define that the control input is given by

$$\hat{u}(t) = \mathcal{K}(t, \Im(t)) \tag{5}$$

where from our previous discussion, $\Im(t) = \{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)}$, and will be called as the *information* available to $\mathcal{C}$ at time t. Later in Section IV, we will notice that for systems where $\tilde{A}$ is not Hurwitz, the controller ($\mathcal{C}$) needs more *information* than the state value $x(t_i)$ at the triggering instances to ensure $\|e\| \le \epsilon$. In fact, we will notice that $\mathcal{E}$ needs to send the pair $(x(t_i), e(t_i))$ to $\mathcal{C}$ at each triggering instance t_i. Therefore, in that case $\Im(t) = \left\{x_0, \{x(t_i)\}_{i=1}^{N(t)}, \{e(t_i)\}_{i=1}^{N(t)}\right\}$.

The event-generator $\mathcal{E}$ is attached to the plant and makes its decision at time t based on the information $\mathcal{F}_t = \{x(s)\}_{0 \le s \le t}$.

It is straight forward to notice that, the more measurements are acquired, the "closer" $x(t)$ will be to $x_c(t)$. For given T and $\epsilon > 0$, the requirement is to design the $(\mathcal{E}, \mathcal{C})$ pair such that $\sup_{t \in [0,T]} \|x_c(t) - x(t)\| \le \epsilon$ for every realization of the disturbance $d(\cdot)$ while minimizing the number of measurements sent by $\mathcal{E}$ to $\mathcal{C}$.

First, we will solve this problem for a finite horizon $[0, T]$ and later we take $T \to \infty$ to study the infinite horizon behavior. Therefore, for a finite horizon $[0, T]$, formally we have the following.

Problem III.1: For any given $\epsilon > 0$

$$\inf_{\mathcal{E},\mathcal{C}} \sup_{d(\cdot) \in \mathcal{D}} N(T) \tag{6}$$

$$\text{s.t.} \sup_{t \in [0,T]} \|x_c(t) - x(t)\| \le \epsilon \quad \forall d(\cdot) \in \mathcal{D} \tag{7}$$

where the admissible policies for the controller are of the form (5) and the admissible policies for the event-generator to compute the triggering sequences $\{t_i\}_{i=1}^{N(t)}$ is such that $N(t) < +\infty$ for all $t \in [0, T]$.

For the infinite horizon case, we consider the following problem (the formulation ensures the optimization problem attains a finite value when there is a feasible event–triggered $\mathcal{E}$ and $\mathcal{C}$).

Problem III.2: For any given $\epsilon > 0$, and some $T < +\infty$

$$J_T^* = \inf_{\mathcal{E},\mathcal{C}} \sup_{d(\cdot) \in \mathcal{D}} \sum_{i=1}^{N(T)} e^{-t_i} \tag{8}$$

$$\text{s.t.} \sup_{t \in [0,T]} \|x_c(t) - x(t)\| \le \epsilon \quad \forall d(\cdot) \in D. \tag{9}$$

Let $\mathcal{E}_T^*$ and $\mathcal{C}_T^*$ be the solution of the finite horizon problem with cost J_T^*. Then, for infinite horizon

$$J_\infty^* = \limsup_{T \to \infty} J_T^* \tag{10}$$

$$\mathcal{E}_\infty^* = \limsup_{T \to \infty} \mathcal{E}_T^* \tag{11}$$

$$\mathcal{C}_\infty^* = \limsup_{T \to \infty} \mathcal{C}_T^*. \tag{12}$$

We would like to mention that $\limsup_{T\to\infty} \mathcal{E}_T^*$ or $\limsup_{T\to\infty} \mathcal{C}_T^*$ is a slight abuse of notation. $\mathcal{E}_T^*$ is characterized by a set that contains the triggering instances and $\mathcal{C}_T^*$ is characterized by the function $\mathcal{K}(t, \Im(t))$. The lim sup is taken over the set of triggering instances and the function $\mathcal{K}(\cdot, \cdot)$.

We assume that Problems III.1 and III.2 are feasible, i.e., for each problem there exists a pair $(\mathcal{E}, \mathcal{C})$ such that the problem has a finite value at the optimum.

In event-triggered control, the design of $\mathcal{E}$ needs to satisfy non-Zeno behavior, i.e., there should not be infinite number of state transmission within any finite time interval. Note that both the problem formulations exclude Zeno behavior: any $\mathcal{E}$ with Zeno behavior will result in an infinite cost. However, for the infinite horizon problem, countable number of triggerings over the horizon $[0, \infty)$ are admissible as long as there is a finite number of triggerings within any finite interval.

A. Assumptions

The following assumptions are carried throughout this paper.

1) The communication link through which the event-generator sends the information to the controller is delay-free, noiseless, and no packets are dropped out.
2) The system parameters (A, B, K) are assumed to be time invariant.
3) The initial information to the controller is $\Im(0) = \{x(0)\}$.

IV. Optimal Controller Synthesis

In this section, we devote our attention to the effects of the controller $\mathcal{C}$ on the error signal $e(t)$.

For simplicity, we restrict ourselves to the space of control strategies, which are dependent only on the latest measurement, and affine functions of the latest measurement, i.e.,

$$\hat{u}(t) = \mathcal{K}(t, \Im(t)) = L(t)x(\theta(t)) + \psi(t) \tag{13}$$

where $L(t)$ and $\psi(t)$ characterize $\mathcal{K}(t, \cdot)$, and $\psi(t)$ does not depend on $x(\theta(t))$. Later we will remove this assumption and consider a general controller as proposed in (5) and show that the affinity assumption does not lose generality (see Theorem IV.6).

Let us note that, with the control given in (13), the state dynamics evolve as

$$\dot{x}(t) = Ax(t) + BL(t)x(\theta(t)) + B\psi(t) + d(t) \tag{14}$$

and the error $e = x - x_c$ has the dynamics

$$\dot{e}(t) = \tilde{A}e(t) + BKx(t) + BL(t)x(\theta(t)) + B\psi(t)$$
$$e(0) = 0.$$

Whenever the context is not ambiguous, we will suppress the time argument of various functions to maintain brevity.

From (14), we can write

$$x(t) = F(t)x(\theta(t)) + \int_{\theta(t)}^{t} \Phi(t, s)(B\psi(s) + d(s))ds$$

where

$$F(t) = \Phi(t, \theta(t)) + \int_{\theta(t)}^{t} \Phi(t, s)BLds.$$

$\Phi(t, s)$ is the state transition matrix corresponding to drift matrix A, i.e., $\frac{\partial \Phi(t,s)}{\partial t} = A\Phi(t, s)$ and $\Phi(s, s) = I$ for all s, t; under of time-invariant assumption, $\Phi(t, s) = e^{A(t-s)}$.

Thus,

$$\dot{e} = \tilde{A}e + B(KF + L)x(\theta(t)) + B\phi + d_1$$

where $\phi(t) = \psi(t) + K\int_{\theta(t)}^{t}\Phi(t,s)B\psi(s)ds$ and $d_1(t) = BK\int_{\theta(t)}^{t}\Phi(t,s)d(s)ds$.

By solving the linear dynamics of e, we can write

$$e(t) = \tilde{\Phi}(t,\theta(t))e(\theta(t)) + \int_{\theta(t)}^{t}\tilde{\Phi}(t,s)d_1(s)ds + f(t)$$

where $\tilde{\Phi}(t,s) = e^{\tilde{A}(t-s)}$. By using the fact that for all $s \in [\theta(t), t]$, $\theta(s) = \theta(t)$, we obtain

$$f(t) = \int_{\theta(t)}^{t}\tilde{\Phi}(t,s)B\big((KF(s) + L(s))x(\theta(t)) + \phi(s)\big)ds. \tag{15}$$

Thus,

$$\begin{aligned}\|e(t)\| &= \|\tilde{\Phi}(t,\theta(t))e(\theta(t)) + \int_{\theta(t)}^{t}\tilde{\Phi}(t,s)d_1(s)ds + f(t)\| \\ &= \|v_1 + v_2\|\end{aligned}$$

where $v_1 = \tilde{\Phi}(t,\theta(t))e(\theta(t)) + f(t)$ and $v_2 = \int_{\theta(t)}^{t}\tilde{\Phi}(t,s)d_1(s)ds$. From the triangle inequality, $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$. The equality holds if v_1 and v_2 are aligned. Note that the vector v_1 depends on the controller parameters and the disturbance realization until the latest triggering instance $\theta(t)$. v_2 depends on the disturbance realization after time $\theta(t)$. Since the disturbance could be any $\mathcal{L}^1([0,T])$ function, one can show that there exists a disturbance realization such that v_2 could be aligned with v_1. Thus,

$$\begin{aligned}\sup_{d\in\mathcal{D}}\|e(t)\| = &\sup_{d\in\mathcal{D}}\|\tilde{\Phi}(t,\theta(t))e(\theta(t)) + f(t)\| \\ &+ \sup_{d\in\mathcal{D}}\left\|\int_{\theta(t)}^{t}\tilde{\Phi}(t,s)d_1(s)ds\right\|.\end{aligned} \tag{16}$$

In general, one would obtain an inequality rather than the equality in (16) when $\mathcal{D}$ is not $\mathcal{L}^1([0,T])$ or $\mathcal{L}([0,\infty))$. A similar analysis with modifications can be carried out when one considers an inequality instead of equality in (16).

The controller $\mathcal{C}$ would aim to minimize the first term $\sup_{d\in\mathcal{D}}\|\tilde{\Phi}(t,\theta(t))e(\theta(t)) + f(t)\|$ since the second term is entirely characterized by the exogenous disturbance. However, the value of $e(\theta(t))$ is unknown[1] to the controller based on the information send by the event-generator $\mathcal{E}$.

Thus, in order to ensure $\|e(t)\| \leq \epsilon$, the best possible strategy for $\mathcal{C}$ would be to minimize $\|f(t)\|$ since

$$\begin{aligned}&\sup_{d\in\mathcal{D}}\|\tilde{\Phi}(t,\theta(t))e(\theta(t)) + f(t)\| \\ &\quad = \sup_{y\in\mathbb{R}^n, \|y\|\leq\epsilon}\|\tilde{\Phi}(t,\theta(t))y + f(t)\| \\ &\quad = \sup_{y\in\mathbb{R}^n, \|y\|\leq\epsilon}\|\tilde{\Phi}(t,\theta(t))y\| + \|f(t)\|.\end{aligned}$$

[1] Since it requires knowledge of $x_c(t)$, which is unavailable as the continuous feedback control is not implemented.

The first equality is due to the fact that $e(\theta) \in \mathbb{R}^n$ is nondeterministic due to $d(\cdot)$ but $\|e(t)\| \leq \epsilon$ for all t. The second equality is due to the fact that, with $\Im(t) = \{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)}$, $f(t)$ does not depend on $e(\theta(t))$.

Thus, in this case, the necessary and sufficient condition for an optimal $f(\cdot)$ is $f(t) = 0$ for all t. Since $\phi(s)$ does not depend on $x(\theta(t))$, then using (15), $f(t) \equiv 0$ is equivalent to

$$KF(t) + L(t) = 0 \tag{17a}$$

$$\phi(t) = 0 \tag{17b}$$

for all t. The following lemma characterizes the $L(t)$ that is able to satisfy (17).

Lemma IV.1: $L(t) = -K\tilde{\Phi}(t,\theta(t))$ satisfies $KF(t) + L(t) = 0$, where $\tilde{\Phi}(\cdot,\cdot)$ is the state transition matrix corresponding to the drift matrix $\tilde{A} = A - BK$.

Proof: Let us substitute $L(t) = -K\tilde{\Phi}(t,\theta(t))$ in the expression of $F(t)$. Thus,

$$\begin{aligned}F(t) &= \Phi(t,\theta(t)) - \int_{\theta(t)}^{t}\Phi(t,s)BK\tilde{\Phi}(s,\theta(t))ds \\ &= \Phi(t,\theta(t)) + \int_{\theta(t)}^{t}\frac{d}{ds}(\Phi(t,s)\tilde{\Phi}(s,\theta(t)))ds \\ &= \tilde{\Phi}(t,\theta(t)).\end{aligned}$$

Hence, $KF(t) + L(t) = 0$ for all t. ■

The following theorem characterizes the optimal controller structure and its behavior.

Theorem IV.2: Under the event-triggering scheme where the only information sent by $\mathcal{E}$ to $\mathcal{C}$ is the sampled state value $\{x(t_i)\}_{i\in\mathbb{N}}$ at the instances $\{t_i\}_{i\in\mathbb{N}}$, and $\mathcal{C}$ uses an affine controller as given in (13), the optimal controller that ensures $\sup_{d(\cdot)\in\mathcal{D}}\sup_{t\in[0,T]}\|e(t)\| \leq \epsilon$ has the following structure:

$$\hat{u}(t) = -Kx_d \tag{18}$$

where for all $t \in [t_i, t_{i+1})$

$$\begin{aligned}\dot{x}_d &= \tilde{A}x_d \\ x_d(t_i) &= x(t_i).\end{aligned}$$

Proof: Since $\hat{u}(t)$ is of the form (13) and satisfies (17), then $\psi(\cdot) \equiv 0$ and $L(t) = -K\tilde{\Phi}(t,\theta(t))$. Therefore, from (13)

$$\hat{u}(t) = -K\tilde{\Phi}(t,\theta(t))x(\theta(t)).$$

The theorem is proved by noting that $x_d(t) = \tilde{\Phi}(t,\theta(t))x(\theta(t))$ and, at the triggering instances, $\theta(t_i) = t_i$. ■

This structure for the controller, (18), was assumed without justification for optimality in earlier works [5] and [6].

Remark IV.3: Comparing the dynamics of x_c and x_d, we notice a "certainty-equivalence" type property in the controller structure, i.e., x_d is an (worst case) estimate of x_c and the control $\hat{u}$ replaces $x_c(t)$ with the estimate.

Therefore, using the optimal controller as described in Theorem IV.2, we obtain

$$\dot{e} = \tilde{A}e + d_1. \tag{19}$$

The stability and boundedness of e is totally determined by the matrix $\tilde{A}$ and the disturbance d_1 (or equivalently d), and not by any parameter of the controller $\mathcal{C}$ and $\mathcal{E}$. The vast majority of the past work is based on the assumption that $\tilde{A}$ is Hurwitz and $\mathcal{D} \subseteq \mathcal{L}^\infty([0,T])$. The assumption $\mathcal{D} \subseteq \mathcal{L}^\infty([0,T])$ implies $\|d(t)\| \leq \bar{d}$ (for some $\bar{d} \geq 0$); and consequently, $\|d_1(t)\| \leq \bar{d}\|BK\| \int_{\theta(t)}^t \|\Phi(t,s)\|ds$.

Thus, it is trivially true that

$$\|e(t)\| \leq \sup_t \|d_1(t)\| \int_0^t \tilde{\Phi}(t,s)ds \leq \beta \sup_t \|d_1(t)\|$$

where $\beta = \int_0^\infty \tilde{\Phi}(t,s)ds < \infty$ for Hurwitz $\tilde{A}$. Maintaining an event-triggering scheme such that $\sup_t \|d_1(t)\| \leq \epsilon/\beta$ will ensure $\|e(t)\| \leq \epsilon$. By noticing that $\|d_1(t)\| \leq \bar{d}\|BK\| \int_{\theta(t)}^t \|\Phi(t,s)\|ds$, it is sufficient to ensure $\int_{\theta(t)}^t \|\Phi(t,s)\|ds \leq \epsilon/(\beta\bar{d}\|BK\|)$. Therefore, under the assumptions of bounded noise and the Hurwitz closed-loop system, the triggering instances can be computed offline by solving $\int_{\theta(t)}^t \|\Phi(t,s)\|ds \leq \epsilon/(\beta\bar{d}\|BK\|)$. An event-triggered control problem with these assumptions has been studied in [5], and the conclusion was similar.

We want to address what can be done when $\tilde{A}$ is not Hurwitz and/or $\mathcal{D} \not\subseteq \mathcal{L}^\infty([0,T])$? As it seems from the dynamics of $e(t)$, with $\tilde{A}$ being not Hurwitz, the error $e(t)$ will grow exponentially and eventually cross the given bound ϵ. Thus, in order to keep the error within any given bound ϵ, what extra information does $\mathcal{E}$ have to transmit to $\mathcal{C}$? One of the main foci of this paper is in addressing these questions, which are presented in the subsequent sections. Moreover, we are also interested in the minimal number of measurement transmission.

To maintain the continuity of the analysis, let us assume that $\mathcal{E}$ can also measure $e(t)$ for all t, and at each triggering instance t_i, $\mathcal{E}$ sends the information $(x(t_i), e(t_i))$ to $\mathcal{C}$. For now, this is an assumption that $\mathcal{E}$ knows $e(t)$ since in practice $\mathcal{E}$ has only the knowledge of $\mathcal{F}_t$ ($\mathcal{F}_t = \{x(s)\}_{0\leq s\leq t}$) and "somehow" it has to compute $e(t)$. Later in this paper we will show how $\mathcal{E}$ can compute $e(t)$ for all t.

At this point, we restrict ourselves (without loss of generality, see Theorem IV.6 for the general result) to the controller structures

$$\hat{u}(t) = L(t)x(\theta(t)) + \psi(t, e(\theta(t))). \tag{20}$$

Using this control input, and after some simplifications

$$e(t) = g_1(t, e(\theta(t))) + g_2(t, x(\theta(t))) + \int_{\theta(t)}^t \tilde{\Phi}(t,s)d_1(s)ds$$

where for any $q \in \mathbb{R}^n$

$$g_1(t,q) = \tilde{\Phi}(t,\theta(t))q + \int_{\theta(t)}^t \tilde{\Phi}(t,s)B\phi(s,q),$$

$\phi(t,q) = \psi(t,q) + K\int_{\theta(t)}^t \Phi(t,s)B\psi(s,q)ds$ and

$$g_2(t,q) = \left[\int_{\theta(t)}^t \Phi(t,s)(B(KF(s) + L(s)))ds\right] q.$$

By using similar optimality argument in deriving (17), we aim to find a controller $\mathcal{C}$ such that $g_1(t, e(\theta(t))) = 0$ and $g_2(t, x(\theta(t))) = 0$ (or minimize $\|g_1(t, e(\theta(t)))\|$, $\|g_2(t, x(\theta(t)))\|$).

Notice that $g_2(t, x(\theta(t)))$ can be made equal to 0 for all t, if $L(t) = -K\tilde{\Phi}(t,\theta(t))$ as stated in Lemma IV.1.

From the expression of $g_1(t,q)$, for a fixed $q \in \mathbb{R}^n$, one can verify that

$$\begin{aligned}\dot{g}_1(t,q) &= \tilde{A}g_1(t,q) + B\phi(t,q)\\ g_1(\theta(t),q) &= q.\end{aligned} \tag{21}$$

Therefore, $g_1(t,q)$ has a linear dynamics where $\phi(t,q)$ is acting as a control input to that system. Therefore, making $g_1(t,q) = 0$ for all t,q, is equivalent of asking whether (A,B) is a controllable pair. Since we have freedom in selecting $\psi(t,q)$, we can control the value of $\phi(t,q)$ by properly selecting $\psi(t,q)$. The following theorem formally states how to bring $g_1(t,q)$ to 0 by proper choice of $\phi(t,q)$.

Theorem IV.4: If (A,B) is a controllable pair, then there exists a $\phi(t,q)$ such that $g_1(t,q) = 0$ for all $t > \theta(t)$. Moreover, such a $\phi(t,q)$ is linear in q.

Proof: Controllability of (A,B) implies the controllability of $(\tilde{A},B)$. For a controllable time invariant linear system, the state is controllable to the zero state in arbitrarily small time. Therefore, $\forall q \in \mathbb{R}^n$ and $\forall \delta > 0, \exists \phi(\cdot,q) : [\theta(t), r] \to \mathbb{R}^m$ such that for all $r \geq \theta(t) + \delta$

$$\tilde{\Phi}(r,\theta(t))q + \int_{\theta(t)}^r \tilde{\Phi}(r,s)B\phi(s,q) = g_1(r,q) = 0. \tag{22}$$

Equation (22) ensures that such a $\phi(\cdot,q)$ is of the form $\phi(s,q) = \gamma(s)q$ for all s,q. One can verify that

$$\gamma(s) = \begin{cases} K\Phi(s,\theta(t))\big(I - a(s)W\big) & -B'\Phi(\theta(t),s)'W \\ & \forall s \in [\theta(t), \theta(t)+\delta] \\ 0 & s > \theta(t)+\delta \end{cases}$$

can ensure $g_1(r,q)$ is 0 for all $r \geq \theta(t) + \delta$, where

$$\begin{aligned}a(s) &= \int_{\theta(t)}^s \Phi(\theta(t),\sigma)BB'\Phi(\theta(t),\sigma)'d\sigma\\ W(\delta) &= [a\big(\theta(t)+\delta\big)]^{-1}.\end{aligned} \tag{23}$$

Since δ can be made arbitrarily small, we can have $g_1(t,q) = 0$ for all $t > \theta(t)$. ■

Although, δ could be made arbitrarily small in Theorem IV.4, we must note that the gain of the proposed controller in Theorem IV.4 depends on $W(\delta)$. The eigenvalues of $W(\delta)$ increases (arbitrary high) as $\delta \to 0$. Thus, from an implementation point of view, δ should have some finite positive value, even though theoretically δ can be arbitrarily small.

As soon as we set δ to have some finite positive value, we have to ensure that within the period $[t_i, t_i + \delta)$ (t_i is any triggering instance) no triggering occurs. This could be trickier since we need to ensure $\sup_{t\in[t_i,t_i+\delta]} \|e(t)\| \leq \epsilon$ while $d(t)$ can take any realization within the period $[t_i, t_i+\delta)$. Thus, it might cause a Zeno effect in the triggering system. A more detailed discussion to tackle this implementation issue is presented in Section VIII. For the analysis further, we will assume that δ is chosen arbitrarily small.

Controller $\mathcal{C}$ has the freedom to select $\psi(t,q)$ and not $\phi(t,q)$ directly. Therefore, we need to ensure there exists a $\psi(t,q)$ for the proposed $\phi(t,q)$ in Theorem IV.4.

Proposition IV.5: For all $t \in [\theta(t), \theta(t)+\delta]$

$$\psi(t,q) = (K\tilde{\Phi}(t,\theta(t)) - B'\Phi(\theta(t),t)'W)q$$

and $\psi(t,q) = K\tilde{\Phi}(t,\theta(t))q \; \forall t > \theta(t)+\delta$ achieves the $\phi(t,q)$ in Theorem IV.4. Where W is defined in (23).

Proof: We start by using the definition of $\phi(t,q)$

$$\phi(t,q) = \psi(t,q) + K\int_{\theta(t)}^{t} \Phi(t,s)B\psi(s,q)ds$$

and let us choose $\psi(t,q) = K\tilde{\Phi}(t,\theta(t))q + \psi_1(t,q)$. Thus,

$$\begin{aligned}\phi(t,q) &= K\Phi(t,\theta(t))q + \psi_1(t,q)\\ &\quad + K\int_{\theta(t)}^{t} \Phi(t,s)B\psi_1(s,q)ds.\end{aligned}$$

Let us now select

$$\psi_1(t,q) = \begin{cases} -B'\Phi(\theta(t),t)'Wq, & \theta(t)+\delta \geq t \geq \theta(t)\\ 0 & t > \theta(t)+\delta \end{cases}.$$

Thus, one can verify that for all $t \in [\theta(t), \theta(t)+\delta]$

$$\begin{aligned}\phi(t,q) &= K\Phi(t,\theta(t))q - B'\Phi(\theta(t),t)'Wq\\ &\quad - K\int_{\theta(t)}^{t} \Phi(t,s)BB'\Phi(\theta(t),s)'Wqds\\ &= K\Phi(t,\theta(t))\left(I - a(t)W\right)q - B'\Phi(\theta(t),t)'Wq\\ &= \gamma(t)q.\end{aligned}$$

Similarly, for $t > \theta(t)+\delta$

$$\begin{aligned}\phi(t,q) &= K\left(\Phi(t,\theta(t)) - \int_{\theta(t)}^{\theta(t)+\delta} \Phi(t,s)BB'\Phi(\theta(t),s)'dsW\right)q\\ &= K\Phi(t,\theta(t))\left(I - a(\theta(t)+\delta)W\right)q = 0\end{aligned}$$

as desired. ■

Therefore, the corrective control input can be expressed compactly as

$$\begin{aligned}\psi(t,e(\theta(t))) &= K\tilde{\Phi}(t,\theta(t))e(\theta(t))\\ &\quad - 1_{t\leq\theta(t)+\delta}B'\Phi(\theta(t),t)'We(\theta(t))\end{aligned}$$

where

$$1_{x\leq y} = \begin{cases} 1 & x \leq y\\ 0 & x > y \end{cases}$$

is an indicator function.

Thus, under the assumption that (A,B) is controllable, we have proved that for any $t \geq \theta(t)+\delta$

$$e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds$$

and since $\delta > 0$ can be made arbitrarily small, we can conclude that for all $t > \theta(t)$

$$e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds. \tag{24}$$

In the analysis so far, we have restricted ourselves to controller of the form

$$\mathcal{K}(t,\mathfrak{I}(t)) = L(t)x(\theta(t)) + M(t)e(\theta(t)).$$

In the following theorem, we show that the optimal controller is indeed of this form and the restriction does not lose generality.

Theorem IV.6: If (A,B) is a controllable pair and the controller $\mathcal{C}$ has information $\mathfrak{I}(t) = (x_0, \{x(t_i)\}_{i=1}^{N(t)}, \{e(t_i)\}_{i=1}^{N(t)})$, then the optimal controller has the following linear form

$$\hat{u} = L(t)x(\theta(t)) + M(t)e(\theta(t))$$

where $L(t) = -K\tilde{\Phi}(t,\theta(t))$, and $M(t) = K\tilde{\Phi}(t,\theta(t)) - 1_{t\leq\theta(t)+\delta}B'\Phi(\theta(t),t)W(\delta)$. $W(\delta)$ is defined in (23) $\forall\delta > 0$.

Moreover by making δ arbitrarily small, $e(t)$ can be controlled to have the value

$$e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds$$

for all $t > \theta(t)$.

Proof: The proof of this Theorem is presented in Appendix A. ■

As a remark from Theorem IV.6, we obtain that the evolution of the error $e(t)$ is reset at each triggering instance, irrespective of whether $\tilde{A}$ is Hurwitz or not. This is only done through the appropriate construction of the corrective component (ψ) in the control, and without this component $e(t)$ will grow exponentially when $\tilde{A}$ is unstable and hence violate the requirement $\|e(t)\| \leq \epsilon$ for any Zeno effect free triggering strategy.

At this point, we have shown that under the assumption that $\mathcal{E}$ can transmit both $x(\theta(t))$ and $e(\theta(t))$, the controller $\mathcal{C}$ can ensure that $\forall t > \theta(t)$, (24) holds. Therefore, the next step would be to determine the triggering instances t_i (hence characterizing $\theta(\cdot)$) such that $\|e(t)\| \leq \epsilon$ is satisfied. Also, we need to ensure that $\mathcal{E}$ can precisely calculate $e(t_i)$ at each triggering instance so that it can communicate it to the controller.

At this point, we focus on how $\mathcal{E}$ would precisely calculate and send $e(t_i)$ to the controller at each triggering instance. In practice, $\mathcal{E}$ has the knowledge $\mathcal{F}_t$ and calculation of $e(t)$ requires the knowledge of x_c, which is not available. From the dynamics of $x(t)$, if the controller's parameters L and ψ in (20) are known to $\mathcal{E}$, then $\mathcal{E}$ can uniquely determine $e(t)$ by observing $x(t)$ only. To see this, let us define a dummy state $x_d(t)$, which follows the dynamics

$$\begin{aligned}\dot{x}_d &= Ax_d + B\hat{u}\\ x_d(\theta(t)) &= x(\theta(t))\end{aligned}$$

where $\hat{u}$ is the input generated by the optimal controller $\mathcal{K}(t,\mathfrak{I}(t))$. If $\mathcal{E}$ knows the structure of the controller, then $\mathcal{E}$ can compute x_d precisely just by observing $x(t)$. Now, if we define a new variable $\Delta(t) = x(t) - x_d(t)$, then $\Delta(t)$ follows the dynamics

$$\dot{\Delta} = A\Delta + d(t) \quad \forall t \neq \theta(t)$$
$$\Delta(\theta(t)) = 0.$$

From the definition of Δ, one can immediately verify that $BK\Delta(t) = d_1(t)$. Thus, in order to know $d_1(t)$, $\mathcal{E}$ needs to monitor $x(t)$ and compute $x_d(t)$. We notice that due to the third assumption that $\mathfrak{I}(0) = \{x(0)\}$, we have $e(0) = 0$. Hence, based on whether the corrective control ψ is used or not, we either have $e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds$ or $e(t) = \int_0^t \tilde{\Phi}(t,s)d_1(s)ds$ due to (24) and (19), respectively. Therefore, in either situation, $e(t)$ can be calculated by the event-generator $\mathcal{E}$. Hence, even in the absence of $x_c(t)$, $e(t)$ can be calculated precisely only from the knowledge of $x(t)$ and computing the dummy state variable $x_d(t)$.

The structure of $\mathcal{C}$ is determined by the matrix function $L(t)$ and the vector function $\psi(t)$. Fortunately, the optimal $L(t)$ is unique and can be computed offline. Therefore, $\mathcal{C}$ does not need to communicate this information to the event-generator. Similarly, ψ has a structure that is uniquely determined by the parameter δ. Therefore, the only information related to $\mathcal{C}$ that $\mathcal{E}$ needs to know is the value of δ chosen by $\mathcal{C}$. To resolve this issue, in the same framework, $\mathcal{E}$ can prescribe a $\delta(t_i)$ for the controller, and send the augmented information $(x(t_i), e(t_i), \delta(t_i))$ to $\mathcal{C}$. Otherwise, if $\mathcal{C}$ selects $\delta(t_i)$ (for (23)) after receiving $(x(t_i), e(t_i))$, then that value of $\delta(t_i)$ needs to be communicated to $\mathcal{E}$; and this requires a bidirectional communication between the controller and the event-generator. Whichever of these two methods are adopted, our next results are going to be invariant of this choice.

In the following analysis, without loss of generality we will assume that $\delta(t_i)$ is chosen arbitrarily small enough, either by $\mathcal{C}$ or by $\mathcal{E}$, at each triggering time t_i such that $t_i + \delta(t_i) < t_{i+1}$. Since the admissible triggering strategy has finite number of triggerings in $[0, T]$, such a $\delta(t_i) > 0$ must exists for each t_i.

Now that we have the optimal controller designed and event generator having the precise knowledge of $e(t)$ for all t, we are ready to study the optimal event-generating scheme by solving Problem III.1.

V. Optimal Event Generator Synthesis

In this section, we will assume that the optimal $\mathcal{C}$ uses a $\psi(\cdot)$ (or $\mathcal{E}$ prescribes a δ arbitrarily small) such that for all $t > \theta(t)$, we have

$$e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds.$$

Later we will remove the assumption that δ is arbitrarily small and consider a δ, which is finite and bounded from below (see the Discussion in Section VIII).

Definition V.1: Two optimization problems are equivalent if an optimal solution of one is an optimal solution for the other, and the corresponding optimal values are the same for both problems.

Let us formulate an unconstrained optimization problem that is equivalent to Problem III.1.

Problem V.2: For any given $\epsilon > 0$, optimize the following:

$$\inf_{\mathcal{E},\mathcal{C}} J_1(\mathcal{C},\mathcal{E}) \tag{25}$$

$$J_1(\mathcal{C},\mathcal{E}) = \sup_{d(\cdot)\in\mathcal{D}} \left(N(T) + \sup_{s\in[0,T]} c^{\epsilon}(\|e(s)\|)\right)$$

where

$$c^{\epsilon}(x) = \begin{cases} 0 & x \leq \epsilon \\ +\infty & x > \epsilon. \end{cases}$$

Proposition V.3: Problem III.1 is equivalent to Problem V.2.

Proof: Note that optimal $\mathcal{C}$ for both problems will be the same based on the discussion in Section IV. Therefore, we will focus on synthesizing $\mathcal{E}$ only.

Let $\mathcal{E}_1$ be an optimal solution for Problem III.1. By finiteness assumption, $J(\mathcal{C},\mathcal{E}_1)$ is finite, and satisfies the constraint $\sup_{[0,T]}\|e(t)\| \leq \epsilon$. As a result, $J_1(\mathcal{C},\mathcal{E}_1) = J(\mathcal{C},\mathcal{E}_1)$.

Let us assume $\mathcal{E}_2$ be an optimal solution of Problem V.2, then $J_1(\mathcal{C},\mathcal{E}_2) \leq J_1(\mathcal{C},\mathcal{E}_1) = J(\mathcal{C},\mathcal{E}_1)$. Since $J_1(\mathcal{C},\mathcal{E}_2)$ is finite, it satisfies the constraint $\sup_{[0,T]}\|e(t)\| \leq \epsilon$; and therefore, it is a feasible solution for Problem III.1, and further $J(\mathcal{C},\mathcal{E}_2) = J_1(\mathcal{C},\mathcal{E}_2)$.

Thus, $J(\mathcal{C},\mathcal{E}_2) = J_1(\mathcal{C},\mathcal{E}_2) \leq J(\mathcal{C},\mathcal{E}_1) \leq J(\mathcal{C},\mathcal{E}_2)$. Hence, $J(\mathcal{C},\mathcal{E}_1) = J(\mathcal{C},\mathcal{E}_2) = J_1(\mathcal{C},\mathcal{E}_1) = J_1(\mathcal{C},\mathcal{E}_2)$. ■

The construction of Problem V.2 is followed by the well-known barrier function method in optimization [33], however, we do not construct a barrier function which is smooth and continuous, such as log-barrier-functions. Instead of following a gradient-based optimization here on the unconstrained objective $J_1(\mathcal{C},\mathcal{E})$, we will adopt a dynamic programming based approach, where the solution of the dynamic program will be the time instances to trigger the events.

Let us define the set

$$S = \{t_1, \ldots, t_l \mid \quad \forall i, t_i < t_{i+1}, t_1 \geq 0, t_l < T, l \in \mathbb{N}\}.$$

S denotes the set of all possible event-triggering strategies. Analogous to S, let us also define

$$S(t) = \{t_1, \ldots, t_l \mid \quad \forall i, t_i < t_{i+1}, t_1 \geq t, t_l < T, l \in \mathbb{N}\}$$

which is the set of all feasible triggering instances after t.

Since $\min_{\mathcal{E},\mathcal{C}} J_1(\mathcal{C},\mathcal{E}) = \min_{\mathcal{E}} J_1(\mathcal{C}^*,\mathcal{E})$ where $\mathcal{C}^*$ is the optimal controller discussed previously, we will suppress the dependency of J_1 (or J) on $\mathcal{C}^*$ in the following analysis.

Let us denote the value function

$$V(t,e) = \inf_{\mathcal{E}\in S(t), e(t)=e} \sup_{d(\cdot)\in\mathcal{D}} \{|\mathcal{E}| + \sup_{s\in[t,T]} c^{\epsilon}(\|e(s)\|)\}$$
$$V(T,e) = 0. \tag{26}$$

By this definition, $V(0,0)$ will be the solution to Problem V.2. Also note that $\|e(t)\| = \|e\| > \epsilon$ results in $V(t,e) = +\infty$.

From the special structure of $c^{\epsilon}(\cdot)$, we can write $\sup_{s\in[t,T]} c^{\epsilon}(\|e(s)\|) = \sup_{s\in[t,r]} c^{\epsilon}(\|e(s)\|) + \sup_{s\in[r,T]} c^{\epsilon}(\|e(s)\|)$ for all $r \in [t,T]$.

Let us assume $\|e(t)\| \leq \epsilon$ and $\{t_1^*, \ldots, t_k^*\} \in S(t)$ be the optimal triggering instances starting at time t. Therefore, by optimality criterion

$$\begin{aligned} V(t,e) &= \inf_{t_1 \geq t, e(t)=e} \{1_{t_1<T} + \sup_{d(\cdot)\in\mathcal{D}} \sup_{s\in[t,t_1]} c^{\epsilon}(\|e(s)\|) \\ &\quad + V(t_1,0)\} \\ &= 1_{t_1^*<T} + V(t_1^*,0). \end{aligned} \tag{27}$$

One may verify that for all $s > t$

$$V(t,0) \geq V(s,0)$$

ensuring that $V(\cdot,0)$ is a nonincreasing function. Let us define

$$\tau^*(t) = \inf_s \left\{ T > s \geq t \,|\, \|e(s)\| = \epsilon \right\}$$

where in our convention $\inf$ over an empty set evaluates to be $+\infty$. Therefore, it must hold that $t_1^* \leq \tau^*(t)$. Also, we have for all $r \leq \tau^*(t)$, $V(r,0) \geq V(\tau^*(t),0)$. Thus, using $r = t_1^*$ one can obtain

$$1_{t_1^*<T} + V(t_1^*,0) \geq 1_{\tau^*(t)<T} + V(\tau^*(t),0). \tag{28}$$

However, since t_1^* is optimal, then for all $s \geq t$

$$1_{t_1^*<T} + V(t_1^*,0) \leq 1_{s<T} + V(s,0) + \sup_{d(\cdot)\in\mathcal{D}} \sup_{r\in[t,s]} c^{\epsilon}(\|e(r)\|).$$

Substituting, $s = \tau^*(t)$

$$1_{t_1^*<T} + V(t_1^*,0) \leq 1_{\tau^*(t)<T} + V(\tau^*(t),0). \tag{29}$$

Combining (28) and (29), we obtain

$$1_{t_1^*<T} + V(t_1^*,0) = 1_{\tau^*(t)<T} + V(\tau^*(t),0). \tag{30}$$

Since the function $1_{\cdot<T} + V(\cdot,0) : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is nonincreasing and $t_1^* \leq \tau^*(t)$, the necessary and sufficient condition for t_1^* to be optimal is $t_1^* = \tau^*(t)$.

Therefore,

$$V(t,e) = \begin{cases} 1 + V(\tau^*(t),0) & \tau^*(t) < T \\ 0 & \tau^*(t) = +\infty \end{cases}.$$

Thus, the triggering strategy is to wait until the error reaches to the value ϵ and then trigger an event in order to "reset" the error. The event-triggering strategy is given in the following Theorem.

Theorem V.4: If t_i is the ith triggering instance, then

$$\begin{aligned} t_{i+1} &= \inf_s \{T > s > t_i \,|\, \|e(s)\| = \epsilon\} \\ t_1 &= \inf_s \{T > s > 0 \,|\, \|e(s)\| = \epsilon\}. \end{aligned} \tag{31}$$

Proof: The proof follows directly from fact that at any time t, the next triggering instance is given as

$$\tau^*(t) = \inf_s \{T > s \geq t \,|\, \|e(s)\| = \epsilon\}.$$

Thus, $t_1 = \inf_s\{T > s \geq 0 | \|e(s)\| = \epsilon\}$, and at ith triggering instance (t_i), the next triggering instance will be

$$t_{i+1} = \inf_s \{T > s > t_i \,|\, \|e(s)\| = \epsilon\}.$$

■

The event-triggering mechanism (31) is well studied in the literature, e.g., [5], [6], [31], and [22]. However, to the best of our knowledge, none of the prior works formally studies the optimality of such a strategy. In the following, we show that the event-triggering strategy is unique and well defined, i.e., the strategy does not exhibit Zeno behavior.

Theorem V.5: The optimal triggering strategy (31) is unique.

Proof: Let $\mathcal{E}_1 = \{t_1^1, t_2^1, \ldots, t_{l_1}^1\}$ and $\mathcal{E}_2 = \{t_1^2, t_2^2, \ldots, t_{l_2}^2\}$ be any two optimal solutions of the dynamic programming (28). Then, by (31), for all $i = 2, 3, \ldots$

$$t_i^1 = \inf_{s\in(t_{i-1}^1,T)} \left\{ s \geq 0 | \sup_{(t_{i-1}^1,s]} \|e(r)\| = \epsilon \right\}$$

and $\sup_{r\in(t_{l_1}^1,T)} \|e(r)\| < \epsilon$. Similarly

$$t_i^2 = \inf_{s\in(t_{i-1}^2,T)} \left\{ s \geq 0 | \sup_{(t_{i-1}^2,s]} \|e(r)\| = \epsilon \right\}$$

and $\sup_{r\in(t_{l_1}^1,T)} \|e(r)\| < \epsilon$.

$$t_1^1 = \inf_{s\in(0,T)} \left\{ s \geq 0 \mid \sup_{(0,s]} \|e(r)\| = \epsilon \right\} = t_1^2.$$

Therefore, by induction-based argument, one can show that $t_i^1 = t_i^2$ for all i and $l_1 = l_2 = l$. ■

Corollary V.6: The optimal cost $J_1(\mathcal{C}^*, \mathcal{E}^*)$ is finite iff there is no Zeno effect in the event triggering mechanism.

Proof: First, let us assume that the optimal triggering mechanism $(\mathcal{E}^*)$ is free of Zeno behavior. Therefore, for the horizon $[0,T]$, there are only finite number of triggering instance, say $\{t_0, t_1, \ldots, t_l\}$. From (31), within each interval $[0,t_1], [t_1,t_2], \ldots, [t_l,T]$, $\|e(t)\| \leq \epsilon$. Hence, $J_1(\mathcal{C}^*, \mathcal{E}^*) = V(0,0) = l < +\infty$.

Now, let us assume that $J_1(\mathcal{C}^*, \mathcal{E}^*) = J < +\infty$. Thus, $\mathcal{E}^*$ and $\mathcal{C}^*$ ensure $\sup_{d(\cdot)\in\mathcal{D}} \|e(t)\| \leq \epsilon$ for all $t \in [0,T]$. Therefore, from Problem V.2, $J = N(T) = |\mathcal{E}|$. Thus, there are only finitely many triggering instances, and hence, the triggering mechanism does not exhibit Zeno behavior. ■

Furthermore, we claim that $J_1(\mathcal{C}^*, \mathcal{E}^*)$ is always finite, and the finiteness of $J_1(\mathcal{C}^*, \mathcal{E}^*)$ is ensured by showing that there exists a time interval of positive measure between any two triggerings.

Theorem V.7: The inter-event times are bounded from below.

Proof: Let t_i be a triggering instance when $(x(t_i), e(t_i))$ was sent to the controller. Thus, for all $t > t_i$

$$e(t) = \int_{t_i}^{t} \tilde{\Phi}(t,s) d_1(s) ds.$$

If t_{i+1} is the next triggering instance, then

$$\epsilon = \left\| \int_{t_i}^{t_{i+1}} \tilde{\Phi}(t_{i+1},s) d_1(s) ds \right\|.$$

Using $d_1(t) = BK \int_{t_i}^{t} \Phi(t,s) d(s) ds$

$$\epsilon = \left\| \int_{t_i}^{t_{i+1}} (\Phi(t_{i+1},s) - \tilde{\Phi}(t_{i+1},s)) d(s) ds \right\|.$$

Thus,

$$\epsilon \leq \Delta(t_i, t_{i+1})$$

where for all $\sigma \geq \tau$

$$\Delta(\tau,\sigma) = \sup_{s\in[\tau,\sigma]} \|(\Phi(\sigma,s) - \tilde{\Phi}(\sigma,s))\| \int_\tau^\sigma \|d(s)\| ds.$$

Note that $\Delta(\tau,\tau) = 0$; and $\Delta(\cdot,\cdot)$ is an uniformly continuous function. Thus, for all $\epsilon > 0$, $\exists \delta > 0$,[2] such that if $\|(\tau_1,\sigma_1) - (\tau_2,\sigma_2)\| < \delta$, then $\|\Delta(\tau_1,\sigma_1) - \Delta(\tau_2,\sigma_2)\| < \epsilon$ for all $(\tau_1,\sigma_1),(\tau_2,\sigma_2)$ such that $\sigma_i \geq \tau_i, i = 1,2$.

Using $\tau_1 = \tau_2 = \sigma_1 = t_i$, for all σ_2 such that $|\sigma_2 - t_i| < \delta$, $\|\Delta(t_i,\sigma_2)\| < \epsilon$.

However, $\epsilon \leq \Delta(t_i,t_{i+1})$. Thus, $|t_{i+1} - t_i| > \delta$. Therefore, for each triggering instance t_i, the next triggering is atleast after δ amount of time. ∎

In Theorem V.7, the existence of the positive interevent duration depends on the fact that the functions $\sup_{s\in[\tau,\sigma]} \|(\Phi(\sigma,s) - \tilde{\Phi}(\sigma,s))\|$ and $\int_\tau^\sigma \|d(s)\|ds$ are uniformly continuous over the compact domain $[0,T] \times [0,T]$. For an infinite horizon, we still need to show that $\exists\delta$ such that for all i, $|t_{i+1} - t_i| > \delta$. In order to do so, we use our initial assumption that the disturbance $d(\cdot)$ is Lebesgue integrable. Thus, $\int_0^\infty \|d(t)dt\| = M < +\infty$, for some positive M. Moreover, instead of taking $T \to +\infty$, we consider these interval $[\alpha T,(\alpha+1)T]$, where $\alpha \in \mathbb{R}_+$, and let α vary to span $\mathbb{R}_+$. Let, t_i be the ith triggering instance such that $\alpha T \leq t_i < (\alpha+1)T$ for some α. Thus, for $(\alpha+1)T > t > t_i$

$$\begin{aligned} e(t) &= \int_{t_i}^t (\Phi(t,s) - \tilde{\Phi}(t,s))d(s)ds \\ \|e(t)\| &= \left\| \int_{t_i}^t (\Phi(t,s) - \tilde{\Phi}(t,s))d(s)ds \right\| \\ &\leq \sup_{s\in[t_i,t]} \|(\Phi(t,s) - \tilde{\Phi}(t,s))\| \left\| \int_{t_i}^t d(s)ds \right\| \\ &\leq M \sup_{s\in[t_i,t]} \|(\Phi(t,s) - \tilde{\Phi}(t,s))\|. \end{aligned}$$

If we denote now $\bar{\Delta}(\tau,\sigma) = \sup_{s\in[\tau,\sigma]} \|(\Phi(\sigma,s) - \tilde{\Phi}(\sigma,s))\|$, then one can verify that $\bar{\Delta}(\tau,\sigma) = \bar{\Delta}(\tau+\alpha T,\sigma+\alpha T)$ for all $\alpha, T \in \mathbb{R}$. Clearly, $\bar{\Delta}(\tau,\sigma)$ is uniformly continuous in the domain $[0,T]\times[0,T]$; and thus, it will be uniformly continuous in any domain of the form $[\alpha T,(\alpha+1)T] \times [\alpha T,(\alpha+1)T]$.

Thus, if t_i and t_{i+1} are two consecutive triggering instances, within the interval $[\alpha T,(\alpha+1)T]$ for some α

$$\begin{aligned} \epsilon &\leq M \sup_{s\in[t_i-\alpha T,t_{i+1}-\alpha T]} \|(\Phi(t_{i+1}-\alpha T,s) - \tilde{\Phi}(t_{i+1}-\alpha T,s))\| \\ &= \bar{\Delta}(\bar{t}_i,\bar{t}_{i+1}) \end{aligned}$$

where $\bar{t}_i = t_i - \alpha T$ and $\bar{t}_{i+1} = t_{i+1} - \alpha T$; thus, $0 \leq \bar{t}_i \leq \bar{t}_{i+1} \leq T$. Since $\bar{\Delta}(\tau,\sigma)$ is uniformly continuous in $[0,T] \times [0,T]$, there must exist a $\delta > 0$ such that for all $|t_i - t_{i+1}| = |\bar{t}_i - \bar{t}_{i+1}| < \delta$, $\bar{\Delta}(\bar{t}_i,\bar{t}_{i+1}) \leq \epsilon/M$. Thus, interevent times are always δ apart for an infinite horizon problem. This is provided as a remark in the following.

[2] δ will depend on ϵ; it should be denoted it by $\delta(\epsilon)$.

Remark V.8: For finite and infinite horizon problems, the interevent times are bounded from the following by some positive δ.

Lemma V.9: For all $T > 0$, $\exists\rho > 0$ such that $\sup_t \|d(t)\| < \rho$ implies there will be no triggering. ∎

Proof: Note that, for all $t \leq T$

$$\|e(t)\| \leq \rho \int_0^T \|\Phi(T,s) - \tilde{\Phi}(T,s)\| ds.$$

Therefore, for all $\rho \leq \frac{\epsilon}{\int_0^T \|\Phi(T,s)-\tilde{\Phi}(T,s)\|ds}$, $\sup_t \|e(t)\| \leq \epsilon$, and hence, there will be no triggering.

Therefore, for the trivial case $d(t) \equiv 0$, there will be no triggering and moreover, $e(t) \equiv 0$. This shows the well known fact that for a deterministic system, any feedback law can be realized with the only information of the initial state.

To summarize, we have proved that for any time invariant controllable linear system, any linear feedback can be replaced by an event-triggered feedback while satisfying the constraint $\sup_t \|e(t)\| \leq \epsilon$ for any $\epsilon > 0$.

VI. Infinite Horizon Design Problem

In this section, we visit the optimal $(\mathcal{E},\mathcal{C})$ design problem for the infinite horizon problem as presented in Problem III.2.

From Section IV, we notice that the controller synthesis does not depend on the horizon $[0,T]$, and neither does it depend on the cost function. Rather, the design is aimed to satisfy the constraint $\|e(t)\| \leq \epsilon$ for all t. Thus, for an infinite horizon problem, the optimal $\mathcal{C}$ will have the same structure. One can formally prove this statement by repeating the analysis done in Section IV, however, we do not present the analysis here to maintain brevity.

In order generate the optimal $\mathcal{E}$, let us construct the equivalent problem as it was done for the finite horizon problem.

Problem VI.1: For any given $\epsilon > 0$

$$\inf_{\mathcal{E},\mathcal{C}} J_{2,T}(\mathcal{C},\mathcal{E}) \tag{32}$$

where

$$J_{2,T}(\mathcal{C},\mathcal{E}) = \sup_{d(\cdot)\in\mathcal{D}} \left\{ \sum_{i=1}^{N(T)} e^{-t_i} + \sup_{s\in[0,T]} c^\epsilon(\|e(s)\|) \right\}.$$

If $\mathcal{E}_T^*$ is the solution of the above-mentioned problem then $\mathcal{E}_\infty^* = \limsup_{T\to\infty} \mathcal{E}_T^*$.

For any finite horizon $[0,T]$, the optimal number of triggering instances $\{t_1^*,t_2^*,\ldots,t_l^*\}$ depends on the horizon T. Let us formally denote

$$\mathcal{T}(T) = \left\{ t_1^*(T), t_2^*(T), \ldots, t_{N(T)}^*(T) \right\}. \tag{33}$$

With slight abuse of notation, by $\limsup_{T\to\infty} \mathcal{E}_T^*$, we basically want to compute $\mathcal{T}(\infty) = \limsup_{T\to\infty} \mathcal{T}(T)$; and then we want to characterize $\mathcal{E}_\infty^*$ by $\mathcal{T}(\infty)$. In general, for an infinite horizon problem, $\limsup_{T\to\infty} N(T)$ converges to infinity in (33). However, if $\limsup_{T\to\infty} N(T)$ is countable then $\limsup_{T\to\infty} \sum_{i=1}^{N(T)} e^{-t_i}$ is finite. On the other hand, if the triggering strategy forms a continuum of triggering instances (i.e., Zeno behavior), then $\limsup_{T\to\infty}\sum_{i=1}^{N(T)} e^{-t_i} = \infty$. Therefore, the optimal set of triggering instances found from this formulation excludes Zeno behavior, but at the same time it allows for countable number of triggerings which are desired for an infinite horizon problem.

Proposition VI.2: Problem III.2 is equivalent to Problem VI.1.

The proof of this proposition is very similar to the proof of Proposition V.3 and, hence, we omit it.

Let us denote a value function for the infinite horizon problem for any arbitrary interval $[0, T]$ as

$$V_T(t,e) = \inf_{\mathcal{E}\in S(t)} \sup_{d(\cdot)\in\mathcal{D}} \left\{ \sum_{i=1}^{|\mathcal{E}|} e^{-t_i} + \sup_{s\in[t,T]} c^\epsilon(\|e(s)\|) \right\}$$

$$V_T(T,e) = 0 \tag{34}$$

and we are interested in $\limsup_{T\to\infty} V_T(0,0)$.

Let $t_1 > t$ be the first element of $S(t)$, then by dynamic programming principle

$$V_T(t,e) = \inf_{t_1\geq t} \left\{ 1_{t_1<T} e^{-t_1} + \sup_{d(\cdot)\in\mathcal{D}} \sup_{s\in[t,t_1]} c^\epsilon(\|e(s)\|) + V_T(t_1,0) \right\}. \tag{35}$$

Clearly, in this case as well, for all $s > t$ we have

$$V_T(t,0) \geq V_T(s,0).$$

Let us define

$$t_1^*(t) = \inf_{s\in[t,T]} \left\{ s \geq t \mid \sup_{[t,s]} \|e(r)\| = \epsilon \right\}.$$

Therefore, by the same argument as for the finite horizon case

$$V_T(t,e) = \begin{cases} e^{-t_1^*(t)} + V_T(t_1^*(t),0), & t_1^*(t) \leq T \\ 0, & t_1^*(t) = +\infty. \end{cases} \tag{36}$$

Thus, the triggering strategy is exactly the same as what we had before, and the strategy does not depend on the horizon T, although the output of the strategy (i.e., number of triggerings) varies with T. One property to note here is, if $\{t_1,\ldots,t_{N(T_1)}\}$ are the optimal time instances for triggering for a horizon $[0, T_1]$, and $\{s_1,\ldots,s_{N(T_2)}\}$ are the optimal time instances for triggering for a horizon $[0, T_2]$ ($T_2 > T_1$), then $N(T_2) \geq N(T_1)$ and $s_i = t_i$ for all $1 \leq i \leq N(T_1)$. This is nothing but the optimality principle.

From (36), one can obtain

$$V_\infty(t,e) \triangleq \limsup_{T\to\infty} V_T(t,e) = e^{-t_1^*(t)} + V_\infty(t_1^*(t),0).$$

The value function $V_T(t,e)$ depends on T; and its value is nondecreasing as T increases. The following theorem ensures that for the optimal $\mathcal{E}_\infty^*$, the value function $V_\infty(t,e)$ attains a finite value for all t and $\|e\| \leq \epsilon$.

Theorem VI.3: The optimal value of the asymptotic case ($T\to\infty$) of Problem VI.1 is finite at $\mathcal{E}_\infty^*$.

Proof: First let us note that for the asymptotic case

$$V_\infty(t,e) = \limsup_{T\to\infty} \inf_{\mathcal{E}\in S(t)} \sup_{d(\cdot)\in\mathcal{D}} \left\{ \sum_{i=1}^{|\mathcal{E}|} e^{-t_i} + \sup_{s\in[t,T]} c^\epsilon(\|e(s)\|) \right\}$$

$$\limsup_{T\to\infty} V_\infty(T,e) = 0 \quad \forall e \tag{37}$$

and the optimal value of the asymptotic problem is $V_\infty(0,0)$.

Due to Remark V.8, one can show that for the infinite horizon case also there exists $\delta > 0$ such that the interevent times are at least δ apart, i.e., $t_{i+1} - t_i > \delta$ for all i.

Therefore, using (37), one can write

$$V_\infty(0,0) = \sum_{i=1}^{k} e^{-t_i^*} + V_\infty(t_k^*,0)$$

where the first k optimal triggering instances are $t_1^*, t_2^*, \ldots, t_k^*$. Using the fact that $t_{i+1}^* - t_i^* > \delta$ for all $i > 0$ and $t_1^* > \delta$, one can obtain $k\delta < t_k^*$. Hence,

$$V_\infty(0,0) \leq e^{-\delta}\frac{1-e^{-k\delta}}{1-e^{-\delta}} + V_\infty(t_k^*,0).$$

Thus, for any kth triggering time t_k^*

$$V_\infty(0,0) \leq e^{-\delta}\frac{1-e^{-t_k^*}}{1-e^{-\delta}} + V_\infty(t_k^*,0). \tag{38}$$

Hence taking $k\to+\infty$, or equivalently $t_k^*\to\infty$

$$V_\infty(0,0) \leq \frac{e^{-\delta}}{1-e^{-\delta}}.$$

∎

Let $\mathcal{E}_\infty$ be an event generator such that within some finite interval $[T_1, T_2]$, they are N number of generated events. Then, clearly by the definition of $J_{2,\infty}$ in (32), $J_{2,\infty}(\mathcal{C}, \mathcal{E}_\infty) \geq Ne^{-T_2}$ for any controller $\mathcal{C}$. On the other hand, by Theorem VI.3, we have $J_{2,\infty}^* = V_\infty(0,0) \leq \frac{e^{-\delta}}{1-e^{-\delta}}$. Thus, any optimal event-generating policy would have only finite number of triggerings within a finite interval. Thus, to summarize, for an infinite horizon problem, the next triggering time at any time t is given by[3]

$$t_1^*(t) = \inf_s \{s \geq t \mid \sup_{r\in[t,s]} \|e(r)\| = \epsilon\}.$$

The triggering strategy is not necessarily a periodic strategy over the infinite horizon.

VII. Simulation Results

In this section, we will illustrate our approach using a system evolving in $\mathbb{R}^2$ with the dynamics

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -2 & 0.1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 0.75\end{bmatrix}u + \begin{bmatrix}1\\ 1\end{bmatrix}d.$$

The designed control is $u = -x_2$. For this simulation, we used d to be a bounded valued disturbance with values in

[3]This could be formally proved by following the steps similar to the ones used to prove Theorem V.4.

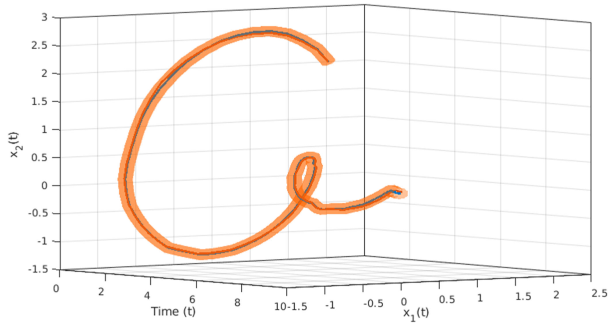

Fig. 2. Behavior of the closed-loop system in shown in red curve and the blue curve shows the same for the event-triggered system. The orange tube has the tolerance radius of 0.1.

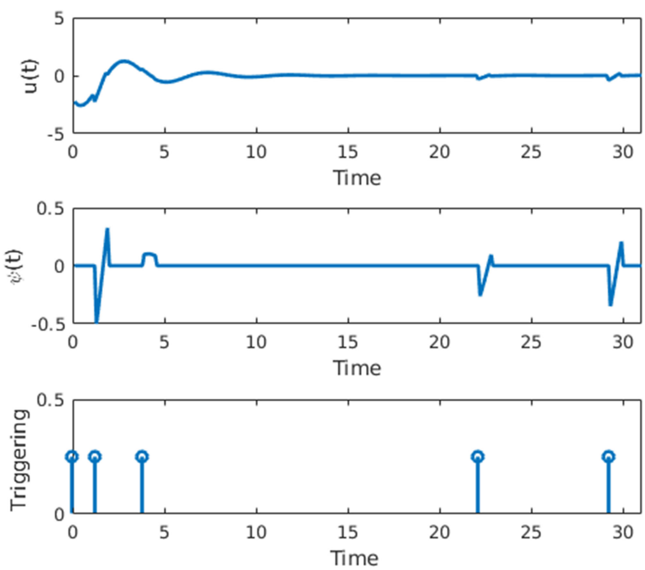

Fig. 3. Top: Control $u(t) = -K\tilde{\Phi}(t,\theta(t))x(\theta(t)) + \psi(t, e(\theta(t)))$, middle: $\psi(t, e(\theta(t)))$, down: Optimal triggering instances.

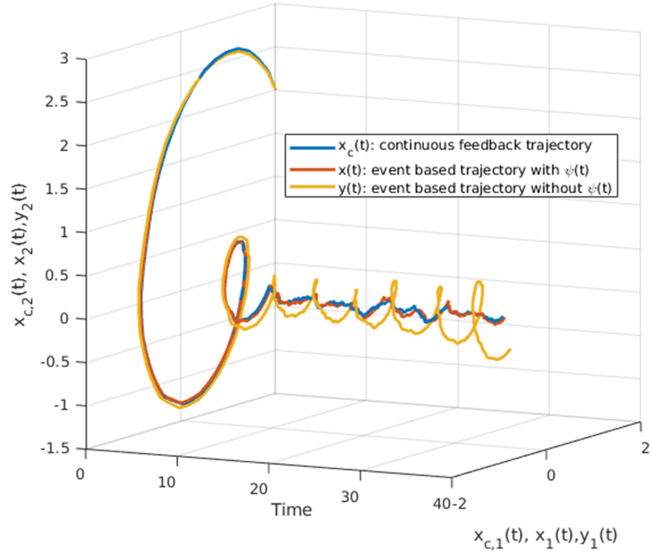

Fig. 4. State trajectories: the closed-loop system, the optimal event-triggered system with information sharing $\mathfrak{I} = \{x(t_i), e(t_i)\}_{i=1,2,\ldots}$, and the event-triggered system with information $\mathfrak{I} = \{x(t_i)\}_{i=1,2,\ldots}$

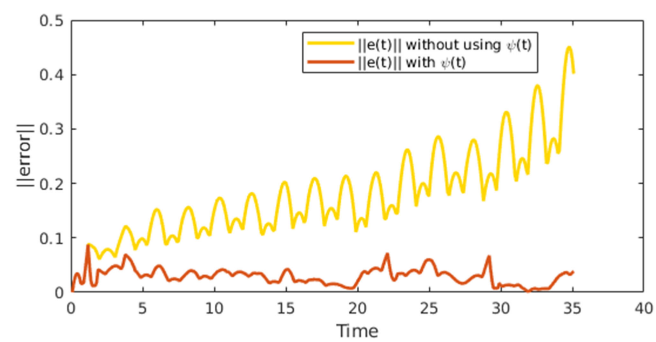

Fig. 5. Norm of the error for event-triggered systems under two different situations: using $\psi(t)$ (red) and without $\psi(t)$ (yellow).

$[-0.5, 0.5]$. The ϵ for this simulation was chosen to be 0.1, and we use $\|\cdot\|_2$ norm, i.e., the requirement is $\|e(t)\|_2 \leq \epsilon = 0.1$ for all t.

In Fig. 2, we show the trajectory of the closed-loop system, the trajectory of the optimal event-triggered system, and the orange tube has a radius of $\epsilon = 0.1$. One may visualize that the phase-plot (projection of the plot in the $x_1 x_2$ plane) is spiral due to the chosen parameters

In Fig. 3, we show the optimal control $u(t)$, $\psi(t)$ and the triggering instances. For the whole time interval of [0, 35], only four (except the one at time 0) triggerings were initiated.

To see the effect of ψ, we performed a simulation under same disturbance and selected $\psi = 0$. The state trajectory and the corresponding error norm is presented in Figs. 4 and 5, respectively.

These figures support the fact that ψ plays a crucial role in ensuring the performance of the system.

VIII. Discussion

A. Note on the Choice of δ in Theorem IV.4

In the analysis, we have theoretically shown that under the controllability assumption on (A, B), the effect of the residual error $e(\theta(t))$ can be nullified in arbitrary small time, and hence, for all $t > \theta(t)$, $e(t)$ does not depend on $e(\theta(t))$. However, as the permitted time (δ in Theorem IV.4) to mitigate the effect of $e(\theta(t))$ gets smaller, the amplitude of the corrective control gets larger; and in the limit $\delta \to 0$, the corrective control component becomes a Dirac delta distribution. In practice, the system might not be able to handle such an "impulsive" nature of the controller. Therefore, in this section, we make an attempt to study the same problem while allowing the controller to have a certain positive amount of time to mitigate this error. The purpose of this section is primarily on the implementation aspect of such a controller, where we aim to show that even without using an "impulsive" controller, the requirement $\|e(t)\| \leq \epsilon$ can still be achieved by slightly changing the threshold for the event-triggering strategy.

In order to do so, we assume that the disturbance is bounded $\sup_t \|d(t)\| \le D$. The study of the problem with arbitrary $d(\cdot)$ is beyond the scope of this paper.

The performance of the (heuristic) method that we are going to propose depends on a parameter $\alpha \in (0.5, 1)$.

First, let us note that by choosing the optimal linear controller we have (see (45) and (47))

$$e(t) = G(t, \Im) + \int_{\theta(t)}^{t} \tilde{\Phi}(t,s) d_1(s) ds$$
$$\dot{G} = AG + Ba$$
$$G(\theta(t), \Im) = e(\theta(t))$$

where $a(t)$ can be chosen freely.

Since (A, B) is a controllable pair, by suitable pole placement $\|G(t, \Im)\| \le e^{-\lambda(t-\theta(t))} e(\theta(t))$ can be achieved for any $\lambda > 0$. Let the event generator $\mathcal{E}$ triggers an event when $\|e(t)\| = \alpha\epsilon$ where $\alpha \in (0.5, 1)$. Let us also note that

$$h(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s) d_1(s) ds = \int_{\theta(t)}^{t} (\Phi(t,s) - \tilde{\Phi}(t,s)) d(s) ds$$

is a differentiable function with $h(\theta(t)) = 0$ and $h'_+(\theta(t)) = 0$.[4]

Let us define

$$D_1 = \sup_s \left\{ s \ge \theta(t) \mid \sup_{r \in [\theta(t), s]} \|h(r)\| \le (1-\alpha)\epsilon \right\}.$$

Due to the above-mentioned properties of $h(t)$, $D_1 - \theta(t)$ is strictly positive, in fact, $D_1 - \theta(t) > \frac{(1-\alpha)\epsilon}{L_h}$ where L_h is the Lipschitz constant of h.

Therefore, for all $t \in [\theta(t), D_1]$, $h(t) \le (1-\alpha)\epsilon$ and we have the following lemma.

Lemma VIII.1: If (A, B) is a controllable pair, then for all $\lambda > 0$ there exists control such that for all $t \in [\theta(t), D_1]$

$$\|G(t, \Im)\| \le \|G(\theta(t), \Im)\| e^{-\lambda(t-\theta(t))}$$
$$G(D_1, \Im) = 0.$$

Moreover, the controller that achieves the above-mentioned requirement, produces control signal of bounded value.

These ensure $\forall t \le D_1$, $\|e(t)\| \le \epsilon$ and at time $t = D_1$

$$\|e(D_1)\| \le \int_{\theta(t)}^{D_1} \|\tilde{\Phi}(t,s) BK d_1(s)\| ds = (1-\alpha)\epsilon < \alpha\epsilon.$$

Thus, the inter-triggering intervals are at least of $(1-\alpha)\epsilon/L_h$ duration, and hence, the controller has $(1-\alpha)\epsilon/L_h$ amount of time to mitigate the effect of the residual error $e(\theta(t))$.

B. Approximate Solution for Optimal Control Problems

Let us consider the following optimal control problem.

[4] h'_+ is the upper-Dini-derivative of h.

Problem VIII.2: Find an event generator and optimal controller pair $(\mathcal{E}, \mathcal{C})$ such that

$$\min_{\mathcal{E},\mathcal{C}} \sup_{d(\cdot)\in\mathcal{D}} \left[N(T) + \alpha \int_0^T l(s, x(s), \hat{u}(s)) ds \right] \tag{39}$$
$$\text{s.t. } \dot{x} = Ax + B\hat{u} + d \tag{40}$$

where $\hat{u}$ is an event-triggered control (linear in measurements) input generated by $(\mathcal{E}, \mathcal{C})$, and $\alpha > 0$.

Solving this problem even for quadratic $l(s, x, u)$ is not trivial (as compared to the well celebrated LQ problems) due to the event-triggered structure of the controller. Such an optimization problem is common when there is a communication cost associated with the transmission of the measurements. However, using our approach one can solve this problem approximately. In the first stage, let us solve Problem VIII.3 to get a linear feedback controller.

Problem VIII.3:

$$\min_{u} \sup_{d(\cdot)\in\mathcal{D}} \left[\int_0^T l(s, x(s), u(s)) ds \right] \tag{41}$$
$$\text{s.t. } \dot{x} = Ax + Bu + d \tag{42}$$

where u is in the space of linear feedback controllers.

Let $u = Kx$ be the optimal solution of Problem VIII.3. In the second stage, the linear feedback control obtained by solving Problem VIII.3 is approximated with an event-triggered control $\hat{u}$ that solves Problem III.1.

Let us denote $u^*(t)$ and $x^*(t)$ to be the optimal control and the corresponding optimal trajectory for the Problem VIII.3. If $\hat{u}(t)$ and $x(t)$ are the optimal approximation of u^* and x^* by solving Problem III.1, then $\|x^*(t) - x(t)\| \le \epsilon$, and one can also show that $\|u^*(t) - \hat{u}(t)\| \le L_u(t)\epsilon$ for some function $L_u(t) > 0$. Thus,

$$\int_0^T l(s, x, \hat{u}) ds = \int_0^T l(s, u^*, x^*) ds + O(\epsilon).$$

Therefore, the event-triggered controller generated in this two-step approach produces a cost which is $O(\epsilon)$ away from the optimal cost of the continuous feedback system. Taking $\epsilon \to 0$, we will have $\int_0^T l(s, x, \hat{u}) ds \to \int_0^T l(s, u^*, x^*) ds$, however, $N(T)$ (the number of triggerings) will be higher as $\epsilon \to 0$.

The study of this optimal control problem is not the aim of this paper, and this section is meant to demonstrate the applicability of this approach beyond the problem described in Problems III.1–VI.1.

IX. Conclusion

In this paper, we propose an optimal controller and event-generator pair to replace a linear continuous feedback controller with an event-triggered one. The deviation in trajectories between the continuous feedback system and the event-triggered system was considered to be a metric of the performance. The analysis shows that for any given performance level $\epsilon > 0$, there always exists a pair of event-triggered controller and event-generator provided the system is controllable.

We show that the optimal controller is linear with respect to the latest information received. The controller also exhibits certainty-equivalence type principle. Moreover, the presence of corrective component $\psi(t, e(\theta(t)))$ is the crucial part of the controller when the closed-loop system is not Hurwitz. It is the $\psi(\cdot,\cdot)$ that ensures that the deviation in the state trajectory does not grow unboundedly for a non-Hurwitz system. Without this component in the controller, the constraint $\|e(t)\| \leq \epsilon$ cannot be guaranteed (as illustrated in the simulation results).

The optimal event-generator follows a threshold strategy where the threshold is the given performance level ϵ. The event generator tracks the error $e(t)$ and transmits the state and error measurements to the controller whenever $\|e(t)\|$ reaches the threshold ϵ. Such a threshold based policy is ubiquitous and easily implementable. Furthermore, such a threshold strategy does not exhibit Zeno behavior.

Appendix

A. Proof of Theorem IV.6

Let us consider the general form of the controller to be

$$\hat{u}(t) = \mathcal{K}\Big(t, \{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)} \cup \{e(t_i)\}_{i=1}^{N(t)}\Big). \tag{43}$$

To maintain brevity we will use $\mathcal{K}(t, \mathfrak{I}(t))$ instead of $\mathcal{K}(t, \{x_0\} \cup \{x(t_i)\}_{i=1}^{N(t)} \cup \{e(t_i)\}_{i=1}^{N(t)})$ where $\mathfrak{I}(t)$ is the information related to the state and error measurements which are available to controller at time t.

Thus,

$$\dot{x} = Ax + B\mathcal{K}(t, \mathfrak{I}(t)) + d \tag{44}$$

which leads to (using the fact that between $\theta(t)$ and t, no information arrives from $\mathcal{E}$ to $\mathcal{C}$, i.e., $\mathfrak{I}(t) = \mathfrak{I}(s) = \mathfrak{I}(\theta(t)) = \mathfrak{I}$ (say) for all $s \in [\theta(t), t]$)

$$x(t) = \Phi(t, \theta(t))x(\theta(t)) + \int_{\theta(t)}^{t} \Phi(t,s)(B\mathcal{K}(s,\mathfrak{I}) + d(s))ds$$

$$x(t) = F(t, \mathfrak{I}) + \int_{\theta(t)}^{t} \Phi(t,s)d(s)ds$$

where $F(t,\mathfrak{I}) = \Phi(t,\theta(t))x(\theta(t)) + \int_{\theta(t)}^{t} \Phi(t,s)B\mathcal{K}(s,\mathfrak{I})ds$.

Therefore, the error signal $e = x - x_c$ can be represented as

$$\dot{e} = \tilde{A}e + BKx + B\mathcal{K}(t,\mathfrak{I})$$

$$\dot{e} = \tilde{A}e + BKF(t,\mathfrak{I}) + B\mathcal{K}(t,\mathfrak{I}) + BK\int_{\theta(t)}^{t} \Phi(t,s)d(s)ds.$$

By denoting $d_1(t) = BK\int_{\theta(t)}^{t} \Phi(t,s)d(s)ds$, we can write $e(t)$ as

$$e(t) = G(t,\mathfrak{I}) + \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds \tag{45}$$

where

$$G(t,\mathfrak{I}) = \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)B(KF(s,\mathfrak{I}) + \mathcal{K}(s,\mathfrak{I}))ds + \tilde{\Phi}(t,\theta(t))e(\theta(t)).$$

Therefore,

$$\begin{aligned}\sup_{d\in\mathcal{D}} \|e(t)\| &= \sup_{d\in\mathcal{D}} \left(\|G(t,\mathfrak{I})\| + \left\| \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds \right\| \right) \\ &= \sup_{d\in\mathcal{D}} \|G(t,\mathfrak{I})\| + \sup_{d\in\mathcal{D}} \left\| \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds \right\|.\end{aligned}$$

All the equalities hold in the above-mentioned derivation since the disturbance could be any function in $\mathcal{L}^1([0,T])$. In fact, one can notice that $G(t,\mathfrak{I})$ depends on the realization of the disturbance until time $\theta(t)$ and the other term $\int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds$ depends on the realization of noise after time $\theta(t)$.

Therefore, in order to keep $\sup_{d\in\mathcal{D}} \|e(t)\|$ as low as possible, controller $\mathcal{C}$ is required to minimize $\|G(t,\mathfrak{I})\|$—by properly selecting $\mathcal{K}(t,\mathfrak{I})$—since the other term ($\| \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)d_1(s)ds\|$) is totally determined by the disturbance and, hence, it is left uncontrolled by $\mathcal{C}$.

To further simplify $G(t,\mathfrak{I})$, we first note that

$$\begin{aligned}&\int_{\theta(t)}^{t} \tilde{\Phi}(t,s)BKF(s,\mathfrak{I})ds \\ &= \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)BK\Phi(s,\theta(t))x(\theta(t)) \\ &\quad + \int_{\theta(t)}^{t} \tilde{\Phi}(t,s)BK\int_{\theta(t)}^{s} \Phi(s,r)B\mathcal{K}(r,\mathfrak{I})drds \\ &= \int_{\theta(t)}^{t} \frac{d}{ds}(\tilde{\Phi}(t,s)\Phi(s,\theta(t)))x(\theta(t)) \\ &\quad + \int_{r=\theta(t)}^{t} \left[\int_{s=r}^{t} \tilde{\Phi}(t,s)BK\Phi(s,r)ds\right] B\mathcal{K}(r,\mathfrak{I})dr \\ &= \Big(\Phi(t,\theta(t)) - \tilde{\Phi}(t,\theta(t))\Big)x(\theta(t)) \\ &\quad + \int_{\theta(t)}^{t} \big(\Phi(t,r) - \tilde{\Phi}(t,r)\big)B\mathcal{K}(r,\mathfrak{I})dr.\end{aligned}$$

Therefore,

$$\begin{aligned}G(t,\mathfrak{I}) = \tilde{\Phi}(t,\theta(t))e(\theta(t)) &+ \big(\Phi(t,\theta(t)) - \tilde{\Phi}(t,\theta(t))\big)x(\theta(t)) \\ &+ \int_{\theta(t)}^{t} \Phi(t,r)B\mathcal{K}(r,\mathfrak{I})dr.\end{aligned} \tag{46}$$

Looking into the expression of $G(t,\mathfrak{I})$ in (46), one can guess that the $\mathcal{K}(t,\mathfrak{I})$, which aims to minimize $\|G(t,\mathfrak{I})\|$, is only a function of $x(\theta(t))$ and $e(\theta(t))$ and moreover, $\mathcal{K}(t,\mathfrak{I})$ has to be linear with respect to $x(\theta(t))$ and $e(\theta(t))$.

Now, we want to check whether there exist matrix valued functions $M(t)$ and $L(t)$ such that $\mathcal{K}(t,\mathfrak{I}) = M(t)e(\theta(t)) + L(t)x(\theta(t))$ can make $\|G(t,\mathfrak{I})\| = 0$ (or arbitrary small).

Let us take $L(t) = -K\tilde{\Phi}(t, \theta(t))$, and this leads to

$$G(t, \Im) = \left(\tilde{\Phi}(t, \theta(t)) + \int_{\theta(t)}^{t} \Phi(t, r)BM(r)dr\right) e(\theta(t)).$$

This is a similar situation as we dealt with in Theorem IV.4. Under the assumption that (A, B) is a controllable pair, we can choose $M(t)$ such a way that $G(t, \Im)$ converges to zero exponentially fast with the decay rate as fast as desired. Moreover, as presented in Theorem IV.4, we can make $G(t, \Im) = 0$ for all $t > \theta(t)$. To see this choose $M(t) = K\tilde{\Phi}(t, \theta(t)) + M_1(t)$, and thus,

$$\begin{aligned} G(t, \Im) &= \left(\Phi(t, \theta(t)) + \int_{\theta(t)}^{t} \Phi(t, r)BM_1(r)dr\right)e(\theta(t)) \\ &= \mathcal{G}(t)e(\theta(t)) \end{aligned}$$

where

$$\begin{aligned} \dot{\mathcal{G}} &= A\mathcal{G} + BM_1 \\ \mathcal{G}(\theta(t)) &= I \end{aligned} \tag{47}$$

where I is the identity matrix. Since (A, B) is controllable, we have that the Grammian

$$W(\delta) = \int_{\theta(t)}^{\theta(t)+\delta} \Phi(\theta(t), s)BB'\Phi(\theta(t), s)'ds$$

is positive definite for all $\delta > 0$. Therefore, by selecting $M_1(t) = -1_{t \le \theta(t)+\delta} B'\Phi(\theta(t), t)'W(\delta)^{-1}$ for all $t \ge \theta(t)$, one can verify that $\mathcal{G}(t) = 0$ for all $t \ge \theta(t) + \delta$. Since δ can be made arbitrarily small, one can conclude $\mathcal{G}(t) = 0$ for all $t > \theta(t)$; and hence, $G(t, \Im) = 0$ for all $t > \theta(t)$.

Thus, $\mathcal{K}(t, \Im)$ is linear with respect to the elements of $\Im$, and furthermore, it only depends on the latest measurements. Therefore, as a result of using such $\mathcal{K}(t, \Im)$, we can conclude that $\forall t > \theta(t)$

$$e(t) = \int_{\theta(t)}^{t} \tilde{\Phi}(t, s)d_1(s)ds.$$

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Dipankar Maity received the B.E. degree in electronics and telecommunication engineering from Jadavpur University, Kolkata, India, in 2013, and the Ph.D degree in electrical and computer engineering from the University of Maryland, College Park, MD, USA, in 2018.

During his Ph.D., he was a visiting scholar with the Technische Universität München, Germany, and the Royal Institute of Technology, Sweden. He is currently a Postdoctoral Fellow with the Georgia Institute of Technology, Atlanta, GA, USA. His research interests include temporal logic based controller synthesis, Control with logical constraints, control with communication constraints, intermittent feedback control, event-triggered control, stochastic games, and integration of these ideas in the context of cyber-physical systems.

John S. Baras received the Diploma degree in electrical and mechanical engineering from the National Technical University of Athens, Athens, Greece, in 1970, and the M.S. and Ph.D. degrees in applied mathematics from Harvard University, Cambridge, MA, USA, in 1971 and 1973, respectively.

Since 1973, he has been with the Department of Electrical and Computer Engineering, University of Maryland, College Park, MD, USA, where he is currently a Distinguished University Professor. He is also a Faculty member of the Applied Mathematics, Statistics and Scientific Computation Program, and an Affiliate Professor with the Fischell Department of Bioengineering, the Department of Mechanical Engineering, and the Department of Decision, Operations and Information Technologies, Robert H. Smith School of Business. Since 2013, he has been a Guest Professor with the School of Electrical Engineering of the Royal Institute of Technology, Stockholm, Sweden. From 1985 to 1991, he was the Founding Director of the Institute for Systems Research (ISR) (one of the first six National Science Foundation Engineering Research Centers). In 1990, he was appointed to the endowed Lockheed Martin Chair in Systems Engineering. Since 1992, he has been the Director of the Maryland Center for Hybrid Networks (HYNET), which he co-founded.

Dr. Baras is a SIAM Fellow, AAAS Fellow, NAI Fellow, IFAC Fellow, AMS Fellow, AIAA Associate Fellow, Member of the National Academy of Inventors (NAI), and a Foreign Member of the Royal Swedish Academy of Engineering Sciences (IVA). Major honors and awards include the 1980 George Axelby Award from the IEEE Control Systems Society, the 2006 Leonard Abraham Prize from the IEEE Communications Society, the 2014 Tage Erlander Guest Professorship from the Swedish Research Council, and a three year (2014–2017) Senior Hans Fischer Fellowship from the Institute for Advanced Study of the Technical University of Munich, Germany. In 2016, he was inducted in the University of Maryland A. J. Clark School of Engineering Innovation Hall of Fame. He was the recipient of the 2017 Institute for Electrical and Electronics Engineers (IEEE) Simon Ramo Medal, the 2017 American Automatic Control Council (AACC) Richard E. Bellman Control Heritage Award, the 2018 American Institute for Aeronautics and Astronautics (AIAA) Aerospace Communications Award, and a Doctorate Honoris Causa by the National Technical University of Athens, Greece, in 2018. He has educated 87 doctoral students, 116 M.S. students and has mentored 55 postdoctoral fellows. He has given many plenary and keynote addresses in major international conferences worldwide. He has been awarded eighteen patents and has been honored worldwide with many awards as innovator and leader of economic development.