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Event Based Control for Control Affine Nonlinear Systems: A Lyapunov Function Based Approach

Dipankar Maity and John S. Baras

Abstract— **In this paper, we propose an event based control strategy for control affine nonlinear systems. The proposed method ensures sufficient reduction in communication by only invoking a communication when some event has occurred. The error between the continuous state feedback nonlinear system and the event based system can be bounded in an invariant set. The upper bound of this error is derived which can be controlled by appropriately choosing the parameters for the event triggering function. This method is then applied to a networked nonlinear system of inverted pendulum and a non-linearizable nonlinear system.**

I. INTRODUCTION

Computing the control law of a large system generally requires continuous reading from the sensors and transmitting these measurements to the control input generators. As a result, the performance depends on the continuous availability of the sensor measurements, and efficient and accurate computation of the control law. For a distributive system, although the computation is done distributively yet it requires continuous information exchange among the subsystems. Communicating the measurements obtained in a sensor network and propagating the data to the controller are a necessary and important part for networked control systems. Consequently, the performance is generally restricted by the available network bandwidth and computing resources.

To overcome the limitation of available communication resources, researchers have come up with novel techniques such as event-based control [1], self triggered control [2], [3] and periodic time control [4], [5] that require only discrete-time communications. These methods ask for discrete communications between the sensor network and the controller, and as a result the controller can only generate a control that approximates the continuous state feedback control. The communication is done periodically, after T amount of time, in periodic control. Finding a suitable time period T to guarantee some level of performance is a main challenge for this approach. In self triggered and event based control, the communication is done only when some event has occurred. Event based control monitors some signal and it triggers for communication based on that signal measurements.

Event based control has attracted a great deal of research in the recent past due to its effectiveness. A study that has been made in [6] on the performance of event based control and periodic control has revealed the fact that under some conditions the event based control performs better than periodic control. Interested readers are directed to confer [7] and [8] and the references therein to get a broad review on event-triggered and self-triggered control, estimation and optimization. Recent publications like [9] considered a state feedback approach for an event based system where the feedback control is generated from another system and this system is updated every time a trigger is generated. In [10] and [11], event based control is proposed for distributed interconnected linear systems.

Despite of the wide applicability of event based control, there is not much work, in literature, for nonlinear event based control. In [12], the authors proposed an event based approach for nonlinear input-to-state-stable (ISS) systems. [13] and [14] studied the event based stabilization of non-linear plants using a Lyapunov based technique. [15] also considered an event based approach for real time scheduling tasks. [16] extended the technique proposed in [15] for homogeneous and polynomial systems. Whereas the previous methods are Lyapunov based approachs, [17] took a different formalism to study event based control for input-output linearizable systems with relative degree equal to the dimension (n) of the statespace. In [18], they refined the method for input-output linearizable input affine nonlinear systems with relative degree $r \leq n$. [17] and [18] focus on the deviation of the event based system from a continuous state feedback system and showed that this error is bounded. However, the rest of the past work mostly focus on the stabilizing behavior of the event based system rather than the actual error incurred due to the event based approach.

In this paper, we also adopt a Lyapunov function based approach for an event based control strategy applied to input affine nonlinear control systems. In many cases, the control input is of state-feedback form to achieve optimality or some other desired performance and hence we consider that the control input is a state-feedback and known to us a priori. Therefore, it will be interesting to see how the system behaves if this control is approximated by a piecewise constant control. In the following sections we are going to explain on how to construct such a piecewise constant approximation of a given control input by using an event-based approach. We show that the error incurred by using this approximated input can be bounded within an invariant subset (Theorem 2.4) and moreover, the volume of this set can be controlled by the choice of some parameters related to the event triggering function. We also show that under the

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The authors are with the Department of Electrical and Computer Engineering and The Institute for Systems Research, University of Maryland, College Park, USA. Email: dmaity@umd.edu, baras@umd.edu

adopted event triggering strategy, there is a minimum time between two successive events (Theorem 3.1)

II. Event-Based Nonlinear Control System

Let us consider the input-affine nonlinear state space model as given in (1).

$$\dot{x} = f(t,x) + \sum_{i=1}^{m} g_i(t,x) \cdot u_i \tag{1}$$

where u_i is the control input and that input is of the form $\gamma_i(x)$ to achieve some desired behavior from the system.

The closed loop system is given in (2)

$$\dot{x}_c = f(t,x_c) + \sum_{i=1}^{m} g_i(t,x_c) \cdot \gamma_i(x_c) \tag{2}$$

We assume the following properties for the nonlinear systems considered in (1) and (2)

(A1) $\gamma_i(\cdot)$ and for all t, $f(t,\cdot)$ and $g_i(t,\cdot)$ are Lipschitz functions with Lipschitz constants L_γ^i, L_f and L_g^i respectively, for all $i = 1, 2, \cdots, m$.

(A2) $f_x(t,x) = \frac{\partial f(t,x)}{\partial x}$ and $((g\gamma)_i)_x(t,x) = \frac{\partial (g\gamma)_i(t,x)}{\partial x}$ are Lipschitz continuous with Lipschitz constants L_1 and L_2^i respectively, where $(g\gamma)_i(t,x) = g_i(t,x)\gamma_i(x)$.

(A3) The closed loop system with $u_i = \gamma_i(x)$ is exponentially stable.

Note that, we do not assume that the system (1) is ISS (Input to state stable), so the stability of (1) with any other input is not guaranteed.

Let us denote

$$F(t,x) = f(t,x) + \sum_{i=1}^{m} g_i(t,x) \cdot \gamma_i(x), \tag{3}$$

and the trajectory of the closed loop system (2) to be $x_c(t)$.

Our aim is to generate the controls u_i in such a way that does not require continuous availability of the state $x(t)$ and the deviation of the trajectory of this event based system from that of (2) with this control is within some tolerance level. To design such a control we will consider $u_i(t) = \gamma_i(x(t_k)),\ \forall t \in [t_k, t_{k+1})$, where $x(t_k)$ is the value of the state at k-th triggering time t_k.

Theorem 2.1: Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = h(t,x)$, where $h : [0,\infty) \times D \to \mathbb{R}^n$ is continuously differentiable, D is some domain in $\mathbb{R}^n$ that contains the origin, and the Jacobian matrix $\partial h/\partial x$ is bounded and Lipschitz on D, uniformly in t.
Let

$$H(t) = \frac{\partial h}{\partial x}(t,x)\bigg|_{x=0}.$$

Then $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system $\dot{x} = H(t)x$.

Proof: For the proof of this theorem, the readers are directed to [19, Theorem 4.15]. ■

Theorem 2.2: The linear system

$$\dot{p} = A(t)p$$

is exponentially stable, where $A(t) = \frac{\partial F}{\partial x}(t,x)\big|_{x=x_c(t)}$.

Proof: Let us consider the system,

$$\dot{(x_c - p)} = F(t, x_c - p) \tag{4}$$

By assumption (A3), (4) is an exponentially stable system and hence $\lim_{t\to\infty}(x_c(t) - p(t)) = 0$. We can write (4) in the following way as given in (5)

$$\dot{x_c} - \dot{p} = F(t,x_c) - \frac{\partial F}{\partial x}(t,x)\bigg|_{x=x_c(t)} p + O(\|p\|^2) \tag{5}$$

where $\lim_{\|p\|\to 0} O(\|p\|^2)/\|p\| = 0$. Since $x_c(t)$ satisfies (2) and by the definition of $A(t)$, we obtain from (5)

$$\dot{p} = A(t)p + O(\|p\|^2). \tag{6}$$

By assumption (A3), both $x_c(t) \to 0$ and $x_c(t) - p(t) \to 0$ exponentially as $t \to \infty$ and as a consequence, $p(t) \to 0$ exponentially as $t \to \infty$. Therefore, $p = 0$ is an exponentially stable equilibrium point for (6). Theorem 2.1 ensures that the linearization of the nonlinear system (6) around $p = 0$ (i.e. $\dot{p} = A(t)p$) is exponentially stable. ■

1) Event Based Closed Loop System and The Error Dynamics: The closed loop system with continuous state feedback is represented in (2). In the event based strategy, since we do not have continuous state feedback, the control law takes the form of $u_i = \gamma_i(x(t_k))$ where $x(t_k)$ is the value of the state at the previous triggering instance t_k. Therefore, basically u_i is a piecewise constant function. The event based closed loop system is obtained in (7).

$$\dot{x} = f(t,x) + \sum_{i=1}^{m} g_i(t,x)\gamma_i(x(t_k)) \qquad \forall t \in [t_k, t_{k+1}). \tag{7}$$

The error e between the actual closed loop system (2) and the event based closed loop system (7) is defined to be $x_c - x$. e follows the nonlinear dynamics given in (8):

$$\dot{e} = F(t,x_c) - F(t,x) + \sum_{i=1}^{m} g_i(t,x)(\gamma_i(x) - \gamma_i(x(t_k))). \tag{8}$$

Our goal will be to keep this error bounded while only using the limited state measurements available at discrete time instances.

Proposition 2.3: *For all $t \geq t_k$, The dynamics of $e(t)$ can be written as,*

$$\dot{e} = A(t)e + \mu(t,x,x_c)e + \sum_{i=1}^{m} g_i(t,x)(\gamma_i(x) - \gamma_i(x(t_k))) \tag{9}$$

where
$\mu(t,x,x_c) = \frac{\partial F}{\partial x}(t,x)\Big|_{x=x_c(t)-\alpha(t)e(t)} - \frac{\partial F}{\partial x}(t,x)\Big|_{x=x_c(t)}$
and $1 \geq \alpha(t) \geq 0$.

Proof: By assumption (A2), $\frac{\partial F}{\partial x}(t,x)$ is a Lipschitz continuous function. Therefore, using Mean Value Theorem,

$$F(t,x_c) - F(t,x) = \frac{\partial F}{\partial x}(t,x)\bigg|_{x=\bar{x}(t)}(x_c(t) - x(t))$$

where $\bar{x}(t) = \alpha(t)x(t) + (1-\alpha(t))x_c(t) = x_c(t) - \alpha(t)e(t)$, $\alpha(t) \in [0,1]$ depends on $x_c(t), x(t)$ and the function $F(\cdot,\cdot)$.

Using (8) along with the application of Mean Value Theorem on $F(t,\cdot)$, we obtain,

$$\dot{e} = A(t)e + \Big(\frac{\partial F}{\partial x}(t,x)\bigg|_{x=\bar{x}(t)} - A(t)\Big)e + \sum_{i=1}^{m} g_i(t,x)(\gamma_i(x) - \gamma_i(x(t_k))). \quad (10)$$

Using the definition of $A(t)$ and defining $\mu(t,x,x_c) = \frac{\partial F}{\partial x}(t,x)\bigg|_{x=\bar{x}(t)} - A(t)$, we obtain (9). ■

From the expression of $\mu(t,x,x_c)$, clearly $\mu(t,x,x_c)|_{e=0} = \mu(t,x_c,x_c) = 0$. Using this fact, the linearization of $\mu(t,x,x_c)e$ around $e=0$ is: $\frac{\partial(\mu e)}{\partial e}\big|_{e=0} = \left(\mu + \frac{\partial \mu}{\partial e}e\right)\Big|_{e=0} = 0$.

Theorem 2.4: There exists $\epsilon > 0$ and subspace $\Omega_\epsilon \subseteq \mathbb{R}^n$ such that for all t, if $\sum_{i=1}^{m} \|g_i(t,x)\gamma_i(x)\|_2 \leq \epsilon$ and $e(0) \in \Omega_\epsilon$, then the error $e(t) \in \Omega_\epsilon$ for all t.

Proof: Let us first denote

$$\delta(t,x,\{t_l\}_{l=0}^{\infty}) = \sum_{i=1}^{m} g_i(t,x)(\gamma_i(x) - \gamma_i(x(t_k))),$$

where $t_k < t \leq t_{k+1}$. We can consider (9) to be a nonlinear system with perturbation, where the perturbation term is $\delta(t,x,\{t_l\}_{l=0}^{\infty})$ and the unperturbed nonlinear system is:

$$\dot{e} = A(t)e + \mu(t,x,x_c)e. \quad (11)$$

Theorems 2.1 and 2.2 along with the fact that linearization of $\mu(t,x,x_c)e$ around $e=0$ is zero ensure that the unperturbed nonlinear system (11) is exponentially stable. Let $V(t,e) = e^T P(t)e$ be a Lyapunov function for the linear system $\dot{e} = A(t)e$. $P(t)$ satisfies the following properties:
1) $P(t)$ is continuously differentiable, symmetric, bounded, positive define matrix; that is, $0 < c_1 I \leq P(t) \leq c_2 I,\ \forall t$.
2) $P(t)$ satisfies the differential equation $\dot{P}(t) = -A^T(t)P(t) - P(t)A(t) - Q(t)$, where $Q(t)$ is continuous, symmetric, positive definite for all t, i.e. $Q(t) \geq c_3 I > 0$.
Considering the same Lyapunov function for the unperturbed system will show that the unperturbed system (11) is exponentially stable if $\|e\| < r$ for some $r > 0$.

Let us consider the same Lyapunov function for the perturbed system and we obtain,

$$\begin{aligned}\dot{V}(t,e) &= \frac{d}{dt}e(t)^T P(t)e(t) \\ &= e^T(A^T(t)P(t) + P(t)A(t) + \dot{P}(t))e \\ &\quad + 2e^T P(t)(\mu(t,x,x_c)e + \delta(t,x,x(t_k))) \quad (12) \\ &= -e^T Q(t)e + 2e^T P(t)(\mu(t,x,x_c)e + \delta(t,x,\{t_l\}_{l=0}^{\infty})).\end{aligned}$$

Using the definition of $\mu(t,x,x_c)$ given in Proposition 2.3 and the Lipschitz continuity assumption, (A2), on $F(t,x)$ we can write $\|\mu(t,x,x_c)\|_2 \leq (L_1 + \sum_{i=1}^{m} L_2^i)\|e\|_2 = L\|e\|_2$. Therefore, from (12), we obtain,

$$\begin{aligned}\dot{V}(t,e) &\leq -c_3\|e\|_2^2 + 2Lc_2\|e\|_2^3 + 2c_2\|\delta\|_2\|e\|_2 \\ &= \|e\|_2(\|e\|_2 - \theta_1)(\|e\|_2 - \theta_2)\end{aligned}$$

where $\theta_1 = \frac{c_3 - \sqrt{c_3^2 - 16Lc_2^2\|\delta\|_2}}{4Lc_2}$, $\theta_2 = \frac{c_3 + \sqrt{c_3^2 - 16Lc_2^2\|\delta\|_2}}{4Lc_2}$.
If θ_1 and θ_2 are real and $\|e\|_2 \in [\theta_1,\ \theta_2]$, then $\dot{V}(t,e) \leq 0$ and hence $\Omega_\epsilon = \{e \in \mathbb{R}^n |\ \|e\|_2 \in [0,\ \theta_2]\}$ is an invariant set. To ensure θ_1, and θ_2 are real, we need $\|\delta\|_2 \leq \frac{c_3^2}{16Lc_2^2}$. By defining ϵ to be $\frac{c_3^2}{16Lc_2^2}$, we have $e(t) \in \Omega_\epsilon$ if $e(0) \in \Omega_\epsilon$ and $\|\delta\|_2 \leq \sum_{i=1}^{m} \|g_i(t,x)\gamma_i(x)\|_2 \leq \epsilon$.
$\|e\|_2$ will be bounded from below by θ_1, however, θ_1 can be made arbitrarily small by controlling $\|\delta\|_2$. ■

Corollary 2.5: Under the same hypothesis as in Theorem 2.4, the behavior of the event based closed loop system (7) remains in a bounded domain around the trajectory of the closed loop system (2).
Corollary 2.5 follows from the fact that $x_c(t) - x(t) = e(t) \in \Omega_e$ and hence $\|x_c(t) - x(t)\|_2 = \|e\|_2 \in [0, \theta_2]$. This implies $x(t)$ remains is a domain $\Omega(x_c) = \{x \in \mathbb{R}^n |\ \|x_c - x\|_2 \in [0,\ \theta_2]\}$.

In Theorem 2.4, we have established an if-then relationship between the perturbation $\delta(t,x,\{t_l\}_{l=0}^{\infty}))$ and the error e. In the next theorem, we will state the exact relationship between the error $e(t)$ and the perturbation $\delta(t,x,\{t_l\}_{l=0}^{\infty})$.

Theorem 2.6: Consider the dynamics given in (9) and suppose that we have a Lyapunov function $V(t,e)$ that satisfies

$$c_1\|e\|_2^2 \leq V(t,e) \leq c_2\|e\|_2^2 \quad (13)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A(t)e + \mu(t,x,x_c)e) \leq -c_3\|e\|_2^2 \quad (14)$$

$$\left\|\frac{\partial V}{\partial e}\right\|_2 \leq c_4\|e\|_2 \quad (15)$$

for all $(t,e) \in [0,\infty) \times \mathbb{R}^n$ for some positive constants c_1, c_2, c_3 and c_4. Then,

$$\|e(t)\|_2 \leq \frac{c_4}{2c_1}\int_0^t e^{-(t-s)c_3/2c_2}\|\delta(s,x,\{t_l\}_{l=0}^{\infty})\|_2 ds \quad (16)$$

Proof: Let us consider the Lyapunov function described in the statement of this theorem and apply it for the system (9). We obtain,

$$\begin{aligned}\dot{V}(t,e) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A(t)e + \mu(t,x,x_c)e + \delta(t,x,\{t_l\}_{l=1}^{m})) \\ &\leq -c_3\|e\|_2^2 + c_4\|e\|_2\|\delta\|_2 \quad (17)\end{aligned}$$

Let $U(t) = \sqrt{V(t,e(t))}$ and whenever $V(t,e(t)) > 0$, we have $\dot{U}(t) = \frac{\dot{V}(t,e(t))}{2\sqrt{V(t,e(t))}}$ and using (13) and (17) we obtain,

$$\dot{U}(t) \leq -\frac{c_3}{2c_2}U(t) + \frac{c_4}{2\sqrt{c_1}}\|\delta\|_2 \quad (18)$$

We have, $V(t,x) = \int_0^x \frac{\partial V(t,y)}{\partial y} dy = \int_0^1 \frac{\partial V(t,sx)}{\partial x} ds x$ and hence using (15), $V(t,x) \le c_4 \int_0^1 s ds \|x\|_2^2 = \frac{c_4}{2}\|x\|_2^2$. This implies $c_4 \ge 2c_1$.
Therefore, when $V(t,e(t)) = 0$ (i.e. $e(t) = 0$), we have $V(t+h, e(t+h)) \le \frac{c_4}{2}\|e(t+h)\|_2^2 = \frac{c_4}{2}\|\delta\|_2^2 h^2 + o(h^2)$; where $\lim_{h\to 0} o(h^2)/h^2 = 0$

$$D^+U(t) = \lim_{h\to 0^+} \frac{\sqrt{V(t+h, e(t+h))}}{h} \le \frac{c_4}{2\sqrt{c_1}}\|\delta\|_2 \quad (19)$$

Therefore, using (18) and (19), we can write,

$$D^+U \le -\frac{c_3}{2c_2}U(t) + \frac{c_4}{2\sqrt{c_1}}\|\delta\|_2 \quad (20)$$

Using Comparison Lemma [19, Section 3.4], we obtain,

$$U(t) \le U(0)e^{-tc_3/2c_2} + \frac{c_4}{2\sqrt{c_1}} \int_0^t e^{(s-t)c_3/2c_2}\|\delta\|_2 ds \quad (21)$$

Since $e(0) = 0$, (21) and (13) ensure,
$\|e(t)\|_2 \le \frac{c_4}{2c_1} \int_0^t e^{-(t-s)c_3/2c_2}\|\delta(s,x,\{t_l\}_{l=0}^\infty)\|_2 ds$ ■

III. Event Triggering Strategy

Since piecewise constant control is used instead of continuous feedback to drive the system (1), the system will fluctuate from its expected behavior. We will implement an event triggering strategy so that the system determines when the exact state $x(t)$ has to be transmitted to the control generator and the behavior of the system does not go beyond the tolerance level. Our goal is to keep $e(t)$ within the given tolerance level. Theorems 2.4 and 2.6 give explicit relation between the error $e(t)$ and the perturbation $\delta(t,x,\{t_l\}_{l=0}^\infty)$ caused by control mismatch.

We consider a simple event triggering function, based on the instantaneous value of $\delta(t,x,\{t_l\}_{l=0}^\infty)$, given in (22):

$$f_{event}(\|\delta(t,x,\{t_l\}_{l=0}^\infty)\|_2) = \epsilon - \|\delta\|_2 \quad (22)$$

where $\epsilon > 0$. Other variants of event triggering functions are possible and we refer to some of them but due to space limitation, we proceed with our further analysis based on the stated event triggering function given in (22). Analysis for other event triggering function such as (23) and (24) are similar and straight forward.

$$f^1_{event}(\|\delta\|_2) = \epsilon_1 + \epsilon_2 e^{-at} - \|\delta\|_2 \quad (23)$$

$$f^2_{event}(\|\delta\|_2) = \epsilon_3 - \int_{t_k}^t e^{-(t-s)c_3/2c_2}\|\delta\|_2 ds \quad (24)$$

where t_k is the time instance when the last event was triggered, $\epsilon_1, \epsilon_2, \epsilon_3$ and a are some design parameters which take nonnegative values.
The $k+1$-th event is generated when f_{event} (similarly f^1_{event} or f^2_{event}) attains a value of zero. The event triggering mechanism (22) was used for event based control in several works like [9] and [20], whereas (23) was used in [20].

If the ϵ defined in (22) is same as that given in Theorem 2.4, we can guarantee that error will be bounded in a positive invariant set Ω_e or equivalently the event based state trajectory will be bounded in a domain around the closed loop state trajectory. For any other arbitrary ϵ, Theorem (2.6) ensures that $\|e(t)\|_2 \le \frac{c_2 c_4}{c_1 c_3}\epsilon$ for all $t \in [0,\infty)$. Therefore, in any case, for all chosen ϵ there exist $r_2(\epsilon) > r_1(\epsilon) \ge 0$, such that $\Omega(x_c) = \{x \in \mathbb{R}^n | \ r_1 \le \|x - x_c\|_2 \le r_2\}$ is an invariant set. Similarly for the event trigger mechanisms (23) and (24), there exist such invariant sets which depend on the choice of the design parameters $\epsilon_1, \epsilon_2, \epsilon_3$ and a.

Theorem 3.1: Inter event time for the event triggering mechanism(s) defined in (22) (or in (23) and (24)) is bounded from below.

Proof: Clearly, due to the fact that $\gamma_i(x)$ is a Lipschitz continuous function, we have

$$\|\delta\|_2 \le L_\gamma \sum_{i=0}^m \|g_i(t,x)\|_2 \|x(t) - x(t_k)\|_2$$

where $L_\gamma = \max\{L^i_\gamma | \ i = 1,2,\cdots,m\}$. Since $g_i(t,\cdot)$ is a Lipschitz continuous function and $x(t)$ remains in a compact domain $(\Omega(x_c))$, $g_i(t,\cdot)$ is bounded for all t. Hence, we can write, $\sum_{i=0}^m \|g_i(t,x)\|_2 \le G_\infty < \infty$.
As a result, $\|x(t) - x(t_k)\|_2 \le \frac{\epsilon}{G_\infty L_\gamma}$ ensures $\|\delta\|_2 \le \epsilon$.

$$\begin{aligned} \|x(t) - x(t_k)\|_2 \le & \\ & \left\| \int_{t_k}^t \left(f(s,x) + \sum_{i=1}^m g_i(s,x)\gamma_i(x(t_k)) \right) ds \right\|_2 \\ & \le \int_{t_k}^t \left(\bar{L}\|x(s) - x(t_k)\|_2 + K(s) \right) ds \quad (25) \end{aligned}$$

where $\bar{L} = L_f + \sum_{i=0}^m L^i_g \|\gamma_i(x(t_k))\|_2$ and $K(s) = \left\| f(s,x(t_k)) + \sum_{i=0}^m \gamma_i(x(t_k)) g_i(s,x(t_k)) \right\|_2$.
Using Gröanwall-Bellman inequality for (25), we get,

$$\|x(t) - x(t_k)\|_2 \le e^{\bar{L}(t-t_k)} \int_{t_k}^t K(s) ds \quad (26)$$

If $t_k + T$ is the time when the $k+1$-th event was triggered, then from (26)

$$e^{\bar{L}T} \int_0^T K(s+t_k) ds \ge \frac{\epsilon}{G_\infty L_\gamma} \quad (27)$$

Defining $\phi(T) = e^{\bar{L}T} \int_0^T K(s+t_k) ds$, we have $\phi(0) = 0$ and $\phi(T)$ is continuous, increasing with finite $\dot{\phi}(T)$ for all $T \in [0,\infty)$. Therefore, there exists $T_{\min} > 0$ such that for all $T < T_{\min}$, $\phi(T) < \frac{\epsilon}{G_\infty L_\gamma}$ and hence, the inter event time is bound from below by $T_{\min}$, where $T_{\min}$ is such that $e^{\bar{L}T_{\min}} \int_0^{T_{\min}} K(s+t_k) ds = \frac{\epsilon}{G_\infty L_\gamma}$. ■

Theorem (3.1) suggests that the event triggering mechanism does not exhibit Zeno behavior [21].

IV. Simulation Results

A. Example 1: Interconnected Inverted Pendulums

The first example demonstrates the application of the event based nonlinear control scheme, presented in this paper, on a network of inverted pendulums (Figure 1). The nonlinear dynamics for each pendulum is given in (28):

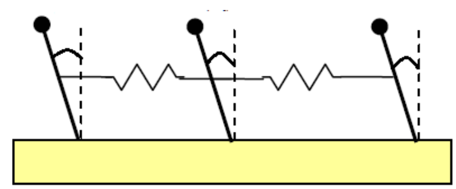

Fig. 1. Three pendulums interconnected by springs. The angular positions are measured anticlockwise from the vertical axes. [12]

$$\dot{x}^i = \begin{bmatrix} x_2^i \\ \frac{g}{l}\sin(x_1^i) - \frac{a_i k}{ml^2}x_1^i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u^i + \begin{bmatrix} 0 \\ \sum_j \frac{h_{ij} k x_1^j}{ml^2} \end{bmatrix} \quad (28)$$

where x_1^i is the angular position of the i-th pendulum and x_2^i is the angular velocity. g is the acceleration due to gravity, l is the length and m is the mass of a pendulum, k is the spring constant and a_i is the number of springs attached to the i-th pendulum. We consider the following parameter values to conduct the experiment: $g = 10, m = 1, l = 2$ and $k = 5$. The parameter values are taken from [12] and [20], where the authors considered the linearlized dynamics of the pendulums to study event based control of networked linear systems. $h_{ij} = 1$ if the i-th and j-th pendulums are connected by a spring, otherwise, $h_{ij} = 0$. Similar to the approach adopted in [12] and [20], we consider the control inputs to be as given below so that the poles of the closed loop system for each pendulum are at -1 and -2.

$$u^i = -(2ml^2 - k)x_1^i - mgl\sin(x_1^i) - 3ml^2 x_2^i - kx_1^2$$

if $i = 1, 3$ and for $i = 2$

$$u^i = -2(ml^2 - k)x_1^i - mgl\sin(x_1^i) - 3ml^2 x_2^i - k(x_1^1 + x_1^3).$$

These control laws linearize the system and decouple each subsystem. A candidate Lyapunov function of the form $(x^i)^T P^i x^i$ for each subsystem can be chosen independently. We choose $P^i = P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which gives $Q = \begin{bmatrix} -4 & -3 \\ -3 & -4 \end{bmatrix}$. Therefore, $c_1 = 0.38, c_2 = 2.62, c_3 = 1$ and $c_4 = 5.24$. We choose the value of ϵ to be 0.05. The initial condition for the system is chosen to be $[\pi/3, 0, -\pi/5, 0, -2\pi/3, 0]$. The behavior of the closed loop system and event based system are plotted in Figure 2.

The errors associated with each dimension are plotted in Figure 3 where the event triggering instances are also shown. Total number of events triggered is 49 and the average interval between two events is 0.4083.

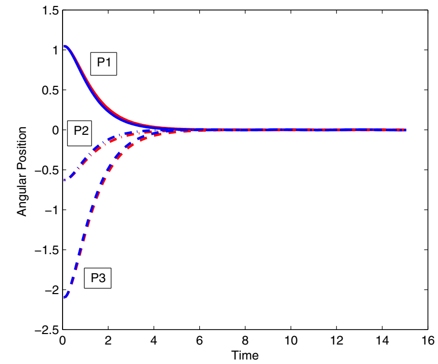

Fig. 2. The red curves show the behavior under continuous feedback and the blue ones for event based feedback. P1, P2 and P3 correspond to pendulum 1, pendulum 2 and pendulum 3 respectively.

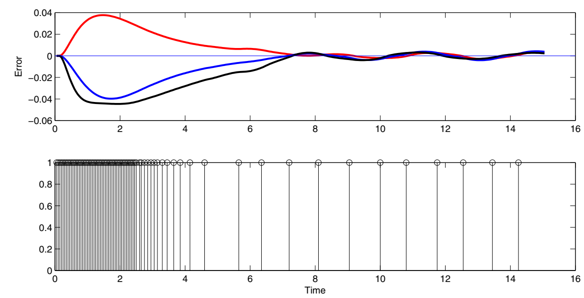

Fig. 3. (upper) The red curve is for the error in angular position of pendulum 1, the blue and black ones are for the same for pendulum 2 and pendulum 3 respectively. (lower) The event triggering profile.

B. Example 2

In the second experiment we consider the following nonlinear dynamics:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sin(x_1) \\ -x_2 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} u \quad (29)$$

This system cannot be linearized for any choice of the control input u but with $u = -x_2$, we can stabilize the system around the origin. A Lyapunov function $V(t, x) = x_1^2 + x_2^2$ proves that the closed loop system is exponentially stable. We choose different initial conditions for this system to observe how the event based system differs from the closed loop system. We choose twelve different initial conditions as shown in Figure 4. The initial conditions are chosen in such a way that two of them lie on the $x_2 = 0$ line and five of them are the reflections of other five about the $x_2 = 0$ axis.

The event triggering profile for the different initial positions are shown in Figure 5. The dynamics is symmetric about the $x_2 = 0$ axis with the chosen control. This symmetry is also reflected in the event triggering pattern. So we only plot the event triggering patterns for the first seven initial conditions shown in Figure 4. The triggering patterns for the 5-th and 7-th initial conditions are same since the initial conditions mirror each other. No further event is triggered after the first one at $t = 0$ for the 6-th and 12-th initial conditions.

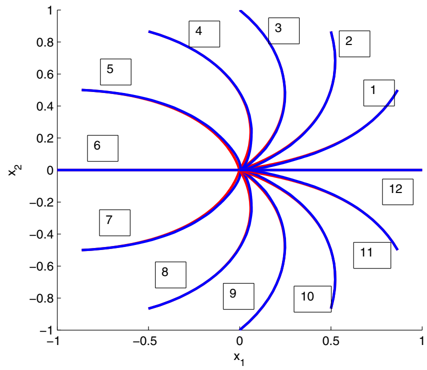

Fig. 4. The red curves are for the continuous feedback system and the blue curves are for the event based system. All the trajectories converge to the equilibrium point at the origin. There are twelve different initial positions as numbered in the figure.

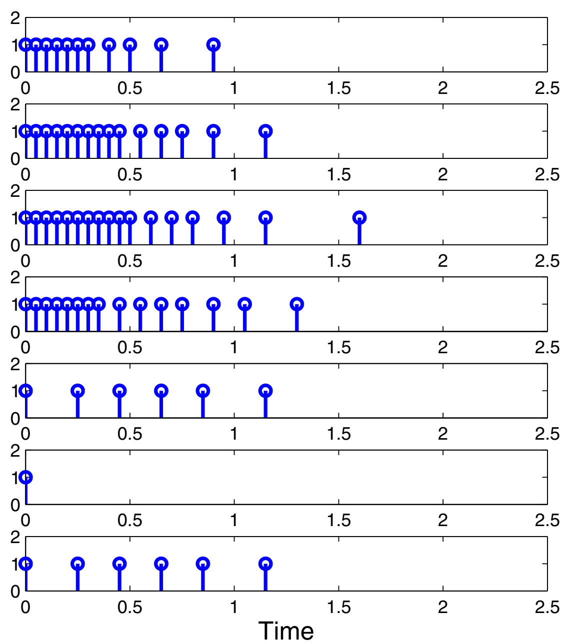

Fig. 5. We only show event triggering profile for the first seven intial conditions, rest of them are similar to their mirroring initial conditions. Intial conditions on the line $x_2 = 0$ do not require any triggering (expect the one at $t = 0$ to set the initial values) because $u \equiv 0$. Event triggering profile is same for intial conditions 5 and 7 since they mirror each other.

V. Conclusions

In this work, we have proposed an event based control strategy for an input affine nonlinear system. We use an event triggering strategy to ensure that the error remains in a bounded domain, and as a consequence, the event based system approximates the behavior of the continuous state feedback system. Simulation results show the application of event based strategy on two input affine nonlinear systems. Theorem 2.6 gives the explicit expression on the boundedness of the error $e(t)$. Theorem 3.1 also shows a relation between the inter event time and the error bound. The optimal error bound can be selected based on the precision needed and the communication resources available for triggering. Possible future works would be to extend this framework to a general nonlinear control systems of the form $\dot{x} = f(t, x, u)$, and include time delays and dropouts into the network.

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